NUMERICAL SOLUTIONS FOR SUB-DIFFUSION PROBLEMS

BY

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This thesis, written by **YOSEPH TIRUWUHA TEREDA** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

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Dedicated to my wonderful mother for her love and measureless support.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

THESIS ABSTRACT

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TITLE OF STUDY: NUMERICAL SOLUTIONS FOR SUB-DIFFUSION PROBLEMS

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In this thesis, we present two numerical methods to solve a class of sub-diffusion equation involving a parameter in the range of $0 < \alpha < 1$*. These numerical methods combine a finite difference in time and linear finite elements in space. The methods use a non-uniform time step to compensate for the non-singular behavior of the exact solution at* $t = 0$*. Stability and convergence of our proposed numerical methods are investigated. In addition to this, numerical experiments are carried out to support our theoretical analysis.*

تشتبف الملخص

الإسم : يوسف تيرووها تيريدا

عنوان الدراسة: الحلول العددية للالمخفتات الفرعية

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في هذه الأطروحة، نقدم اثنين من الطرق العددية لحل فئة من المعادلات الدون نشرية ذات معامل ما بين الصفر والواحد. هذه الطرق العددية تدمج بين الفرق المحدود للزمان والعناصر المحدودة الخطية في الفضاء. هذه الطرق تستخدم خطوات وقتية غير موحدة للتعويض عن السلوك الشاذ للحل الصحيح عند بداية الوقت. لقد تم استقصاء الاستقرار والتقارب للطرق العددية المقترحة، كما تم اجراء تجارب عددية لدعم تحليلنا النظري.

CHAPTER 1

INTRODUCTION

1.1 Motivation

Diffusion is the net movement of molecules or atoms from a region of high concentration (or high chemical potential) to a region of low concentration (or low chemical potential).

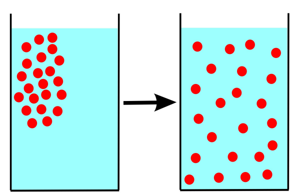


Figure 1.1: *Particle diffusion in fluid*

Anomalous diffusion is a term used to describe a diffusion process with a non-linear relationship to time. Therefore, the means square displacement proportional to τ^α, i.e.,

$$< r^2(\tau) > \quad \propto \quad \tau^\alpha,$$

where α is the exponent of the anomalous diffusion. For the particular case $\alpha = 1$ we have the classical diffusive motion. The motion is super-diffusion in the case $\alpha > 1$ and sub-diffusion in the case $0 < \alpha < 1$.

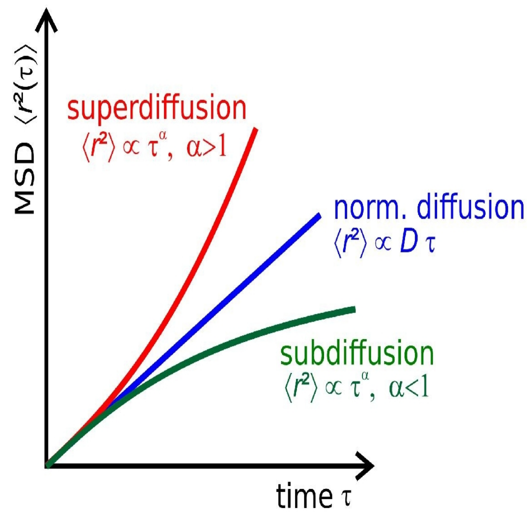


Figure 1.2: *Mean squared displacement $\langle r^2(\tau) \rangle$ for different types of anomalous diffusion.*

Anomalous diffusion is a new and exciting area of research because it is a widespread natural phenomenon. The role of anomalous diffusion has received attention within the literature to describe many physical scenarios, most prominently within crowded systems, for example protein diffusion within cells, or diffusion through porous media, for examples, diffusion through porous media, or protein diffusion within cells. Anomalous diffusion cannot be described by standard tools like diffusion equation. Instead it requires the use of fractional partial differential equations involving fractional derivatives of non-integer order.

In this thesis we are interested in sub-diffusion. Sub-diffusion is observed in motion of lipids on membranes, trans location of polymers, solute transport in porous media, etc. Mostly it occurs when a particle gets trapped and needs a long period of time to get free.

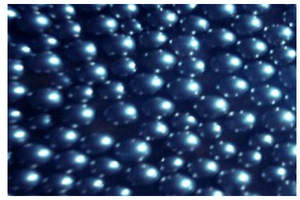


Figure 1.3: *Granular particles.*

1.2 Outline of Thesis

In chapter 1, we introduce the mathematical problem, and set up some notations that will be used throughout this work. In chapter 2, we solve the given problem using separation of variables and Laplace transformation. In addition, we show the stability of the exact solution. In chapter 3, we introduce time discretization, and formulate finite differences method for the given problem. In addition, the stability and convergence of the finite difference approximation are analyzed respectively. Moreover, numerical experiments carried out to support our theoretical results. In chapter 4, introduce spatial discretization, and formulate finite element method for the given problem. In addition, stability and convergence studied in details. In chapter 5, we introduce full discretization, and formulate by combine finite difference with finite element method for the given problem. In addition, we combine the separate estimates for the space and time discretization to bound the error for fully discrete solutions. Moreover, numerical experiments carried out to support our theoretical results.

1.3 Problem Setting

Let Ω is a convex, bounded and polyhedral domain of $\mathbb{R}^n (n = 1, 2)$. For $T > 0$, we consider the problem of solving for $u : \Omega \times (0, T) \longrightarrow \mathbb{R}^n$, and the following sub-diffusion problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + D^\alpha A u = f(x,t), &\qquad \text{in} \quad \Omega \times (0, T], \\ u(x, 0) = u_0(x), &\qquad \text{on} \quad \Omega, \\ u(x,t) = 0, &\qquad \text{on} \quad \partial\Omega \times [0, T], \end{aligned} \tag{1.1}$$

The operator D^α represents a time fractional derivative defined as:

$$D^\alpha v(t) = \frac{\partial}{\partial t} I^{1-\alpha} v(t),$$

where

$$I^{1-\alpha} v(t) = \int_0^t \omega_\alpha(t-s) u(s) ds, \text{ with } \omega_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

I^α denote the Riemann-Liouivlle fractional integral.

Here, $f : \Omega \times (0, T) \longrightarrow \mathbb{R}$ stands for source term, $u_0 : \Omega \longrightarrow \mathbb{R}^n$, is the initial condition and $0 < \alpha < 1$. We take $A := -\nabla.(a\nabla)$ through out the thesis where the coefficient $a = a(x)$ is smooth with $a(x) \geq a_0 > 0$.

Introducing the bilinear form

$$B(w, v) = \langle Aw, v\rangle = \langle a\nabla w, \nabla v\rangle \text{ for } w, v \in H_0^1(\Omega).$$

1.4 Literature Review

Mustupha and Mclean [11] employed a piecewise-constant, discontinuous Galerkin method for the time discretization of a sub-diffusion equations. They proved a prior error bound of order k under realistic assumption on the regularity of the solution. They also showed that a spatial discretization using continuous, piecewise-linear finite elements leads to an additional error term of order $h^2 \max(1, \log k^{-1})$.

Yuste and Acedo [22] proposed an explicit finite difference (FD) method for solving fractional diffusion equation. An $O(k + h^2)$ truncation error was shown assuming that u is sufficiently smooth at $t = 0$. Chen et al. [1] considered the problem and solved by using the Grunwald-Letinikov expansion for time and finite difference method for space and he proved the convergence of order $O(k + h^2)$.

Cui [3] proposed high-order compact finite difference scheme (After approximating the second-order derivative with respect to space by the compact finite difference, they used the Grunwald-Letnikov discretization of the Riemann-Liouville derivative to obtain a fully discret implicit scheme) and he proved the method has accuracy of four in the spatial grid size and one in the fractional time step, provided u is sufficiently smooth.

A compact ADI scheme was studied recently in [4]. This method is used to split the original problem into two separate one-dimensional problems. The local truncation error was analyzed and the stability was discussed by the Fourier method.

The numerical solution of sub-diffusion problems has been extensively studied over the last few decades. For explicit and implicit Euler finite difference (FD) schemes, see the work [2, 3, 8, 9, 13, 20, 22, 24, 25]. For alternating direction implicit FD schemes on a rectangular spatial domain: we refer to [23, 26]. In addition, various numerical methods [4, 5, 6, 12, 15, 16, 23] have also been applied for the following alternative representation of (1.1) (using the caputo dervatives):

$$I^{\alpha}u^{'}(t) - \nabla.(K_{1-\alpha}\nabla u)(t) = f(t).$$

The two representations are equivalent under suitable assumptions on the initial data, but the methods obtained for each representation are formally different.

1.5 Functional and Sobolev Space

For $1 \leq p \leq \infty$, let $L_p(\Omega)$ denote the set of real-valued Lebesgue measurable functions w defined on Ω such that $|w|^p$ is integrable on Ω with respect to the Lebesgue measure $dx = dx_1...dx_n$. We define the L_p norm, $\|.\|_{L_p(\Omega)}$, by

$$\|w\|_{L_p(\Omega)} = \Big(\int_{\Omega} |w(x)|^p dx\Big)^{\frac{1}{p}} \text{ for } p \geq 1.$$

For real-valued functions $w, v \in L_2(\Omega)$, we define the L_2 inner product and the associated norm by

$$\langle w, v\rangle = \int_\Omega w(x)v(x)dx \text{ and } \|w\|_{L_2(\Omega)} = \Big(\int_\Omega |w(x)|^2 dx\Big)^{\frac{1}{2}}.$$

Then for each $w, v \in L_2(\Omega)$, the Cauchy-Schwartz inequality states that $wv \in L_2(\Omega)$, and

$$|\langle w, v\rangle| \leq \|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.$$

For a positive integer s, the Sobolev space $H^s(\Omega)$ is defined as:

$$H^s(w) = \{w \in L_2(\Omega) : D^m w \in L_2(\Omega) \text{ for } 0 \leq |m| \leq s\},$$

where

$$D^m = \frac{\partial^m v}{\partial x_1^{m_1} \ldots \partial x_1^{m_d}} \quad w \in C^k(\Omega) \text{ with } |m| \leq k,$$

and

$$|m| = \sum_{i=1}^{d} m_i.$$

Clearly, $H^s(\Omega)$ is a subspace of $L_2(\Omega)$ and $H^0(\Omega) = L_2(\Omega)$. On $H^s(\Omega)$, we define the inner product

$$\langle w, v\rangle = \sum_{|m|\leq s} \langle D^m w, D^m v\rangle, \forall w, v \in H^s(\Omega).$$

and the associated norm is defined as follows:

$$\|w\|_{s,\Omega} = \left(\sum_{|m| \leq s} \|D^m w\|_{\Omega}^2 \right)^{\frac{1}{2}}, \forall w \in H^s(\Omega).$$

A comprehensive presentation of these spaces can be found in Adams [1] .

CHAPTER 2

WELL-POSEDNESS OF THE CONTINUOUS SOLUTION

In this chapter the stability of a continuous solution of the model problem (1.1) will be investigated first. Then, a series representation solution for (1.1) will be constructed.

2.1 Stability

Stability: The solution must be stable in the sense that it depends continuously on the data.

Lemma 2.1 *If* $u_0 \in L_2(\Omega)$ *and* $f \in L_1((0,T); L_2(\Omega))$*, then the solution* u *of* (1.1) *is stable and belongs to* $L_2((0,T); L_2(\Omega))$*.*

Proof. Taking the inner product of (1.1) with u,

$$\langle u_t, u(t)\rangle + B(D^\alpha u(t), u(t)) = \langle f(t), u(t)\rangle.$$

Since

$$\langle u_t(t), u(t)\rangle = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2.$$

Then by the Cauchy-Schwartz inequality,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + B(D^\alpha u(t), u(t)) \leq \|f(t)\|\|u(t)\|. \tag{2.1}$$

Integrate equation (2.1) from 0 to T yields

$$\|u(T)\|^2 + 2\int_0^T B(D^\alpha u(s), u(s))ds \leq \|u(0)\|^2 + 2\int_0^T \|f(s)\|\|u(s)\|ds.$$

Since by Theorem A.1 [16],

$$\int_0^T B(D^\alpha u(s), u(s))ds \geq 0.$$

We have

$$\|u(T)\|^2 \leq \|u(0)\|^2 + 2\int_0^T \|f(s)\|\|u(s)\|ds.$$

Setting

$$\|RHS(t)\| = \|u(0)\|^2 + 2\int_0^t \|f(s)\|\|u(s)\|ds.$$

Then, we have

$$\frac{d}{dt}\|RHS(t)\| = 2\|f(t)\|\|u(t)\| \leq 2\|f(t)\|\sqrt{\|RHS(t)\|},$$

and so

$$\frac{d}{dt}\sqrt{\|RHS(t)\|} \leq \|f(t)\|.$$

This implies that

$$\sqrt{\|RHS(t)\|} \leq \sqrt{\|RHS(0)\|} + \int_0^t \|f(s)\|ds,$$

So that

$$\|u(T)\| \leq \|u(0)\| + \int_0^T \|f(s)\|ds.$$

Therefore, the proof is completed now. ∎

2.2 A series representation solution and Regularity

The operator $A := -\nabla.(a\nabla)$ subject to homogenous boundary conditions possess a complete eigensystem $\{\lambda_m, \phi_m\}_{m=1}^{\infty}$ in $L_2(\Omega)$ where $\phi_1, \phi_2, \phi_3 ...$ are eigenfunctions and $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$ are the corresponding eigenvalues.

For instance, if $A = -\frac{d^2}{dx^2}$ subject to homogenous boundary conditions on $[0, 1]$

the eigenfunctions and eigenvalues are:

$$\phi_n(x) = \sin(n\pi x) \text{ and } \lambda_n = n^2\pi^2 \text{ for } n = 1, 2, 3...$$

Next, by using the above assumption on A, we have the following series representation:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(t)\phi_m(x) \text{ where } \ u_m(t) = \langle u(t), \phi_m \rangle.$$

and

$$f(t) = \sum_{m=0}^{\infty} f_m \phi_m \text{ where } f_m(t) = \langle f(t), \phi_m \rangle.$$

Taking the inner product of (1.1) with ϕ_m gives a scalar initial -value problem.

$$\frac{du_m}{dt} + \lambda_m(\omega_{1-\alpha} * u_m)_t = \langle f(t), \phi_m \rangle. \tag{2.2}$$

We construct the solution u_m using the Laplace transform

$$\hat{w}(z) = \mathscr{L}\{w(t)\} = \int_o^{\infty} e^{-zt} w(t) dt.$$

Apply Laplace transform to both sides in (2.2), we get

$$z\hat{u}_m(z) - u_{0_m} + \lambda_m z^{\alpha} \hat{u}_m(z) = \hat{f}_m(z),$$

and we have

$$\hat{u}_m(z) = \frac{u_{0_m} + \hat{f}_m(z)}{z + \lambda_m z^{\alpha}}.$$

A geometric series expansion shows that, for any constant $\lambda > 0$,

$$\mathscr{L}^{-1}\left\{\frac{1}{z+\lambda z^{\alpha}}\right\} = E_{1-\alpha}(-\lambda t^{1-\alpha}),$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(t) = \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(1+\alpha p)} \text{ for } 0 < \alpha < 1.$$

Therefore,

$$u_m(t) = u_{0_m} E_{1-\alpha}(-\lambda_m t^{1-\alpha}) + \int_0^t E_{1-\alpha}(-\lambda_m (t-s)^{1-\alpha}) f_m(s) ds.$$

Consequently, for convenience, we introduce the following formula

$$u(t) = \varepsilon(t) u_0 + \int_0^t \varepsilon(t-s) f(s) ds,$$

where

$$\varepsilon(t) w = \sum_{m=0}^{\infty} E_{1-\alpha}(-\lambda_m t^{1-\alpha}) \langle w, \phi_m \rangle \phi_m.$$

Throughout our error analysis, we may use the following regularity property of the continuous solution of problem (1.1),

$$t^{1-\alpha}\|Au^{'}(t)\| + t^{2-\alpha}\|Au^{''}(t)\| \leq Mt^{\sigma-1} \text{ for } t > 0. \tag{2.3}$$

for some $\sigma, M > 0$. For instance, if $f = 0$ and $u_0 \in H^r(\Omega) \cap H_0^1(\Omega)$ for some $r > 0$ then (2.3) holds with $\sigma = (1-\alpha)r$ (see [17]). Due to the lack of regularity of the solution u of (1.1), non uniform time mesh that concentrates the time steps near $t = 0$ will be used.

CHAPTER 3

TIME DISCRETIZATION

In this chapter we semi-discretize the model problem (1.1) in time, using finite difference methods. Stability and convergence will be studied in details.

3.1 Grid points

To define a semi-discrete time-stepping numerical solution of equation (1.1), we first define a grid points in time as follows:

$$t_n = (nk)^\gamma \text{ for } 0 \leq n \leq N, \text{ with } k = \frac{T^{\frac{1}{\gamma}}}{N}, \tag{3.1}$$

$$\text{for some } \gamma \geq 1 \text{ and } N \text{ is the number of intervals,}$$

and

$$k_n = t_n - t_{n-1}, k = \max_{1 \leq n \leq N}(k_n) \text{ and } \; t_{n-\frac{1}{2}} = \frac{1}{2}(t_{n-1} + t_n).$$

Lemma 3.1 *The equation* (3.1) *satisfy the following properties*

$$k_n \leq k_{n+1}, k_n \leq \gamma k t_n^{1-\frac{1}{\gamma}} \text{ for } \; n \geq 1.$$

and

$$t_n \leq 2^{\gamma} t_{n-1} \text{ for } n \geq 2.$$

Proof.

The proof of the first inequality is obvious. For the second inequality, noting first

$$\begin{aligned} k_n &= t_n - t_{n-1} \\ &= \left(\frac{n}{N}\right)^{\gamma} T - \left(\frac{n-1}{N}\right)^{\gamma} T = T\gamma \int_{\frac{n-1}{N}}^{\frac{n}{N}} x^{\gamma-1} dx. \end{aligned}$$

However, $x^{\gamma-1} \leq (\frac{n}{N})^{\gamma-1}$ on $[\frac{n-1}{N}, \frac{n}{N}]$ for $\gamma \geq 1$, then

$$\int_{\frac{n-1}{N}}^{\frac{n}{N}} x^{\gamma-1} dx \leq \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left(\frac{n}{N}\right)^{\gamma-1} dx = \frac{1}{N}\left(\frac{n}{N}\right)^{\gamma-1}.$$

Therefore,

$$k_n \leq \frac{T\gamma}{N}\left(\frac{n}{N}\right)^{\gamma-1} = \gamma k \left(\frac{n}{N}\right)^{\gamma} T \left(\frac{n}{N}\right)^{-1} T^{\frac{-1}{\gamma}} = \gamma k t_{n-1}^{1-\frac{1}{\gamma}}.$$

Since

$$t_n = \left(\frac{n}{N}\right)^{\gamma} T \text{ and } t_{n-1} = \left(\frac{n-1}{N}\right)^{\gamma} T,$$

$$\frac{t_n}{t_{n-1}} = \left(\frac{n}{n-1}\right)^{\gamma} = \left(1 + \frac{1}{n-1}\right)^{\gamma} \leq (1+1)^{\gamma},$$

then we have

$$t_n \leq 2^\gamma t_{n-1} \text{ for } n \geq 2.$$

$\blacksquare$

3.2 Backward Euler method

In this section, the time stepping Backward Euler scheme for the model problem (1.1) will be introduced. The proof of the stability and convergence of this scheme will be discussed.

The Backward Euler scheme is an implicit scheme. We introduce the the function v defined on the time grid points by:

$$\vec{v}(t) = v^n \text{ for } \; t_{n-1} < t < t_n, \quad v^n = v(t_n) \text{ and } \; 1 \leq n \leq N.$$

3.2.1 Scheme

Integrate (1.1) from t_{n-1} to t_n. This gives

$$u^n - u^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (Au)(t)dt = \int_{t_{n-1}}^{t_n} f(t)dt. \tag{3.2}$$

To define Backward Euler solution $U^n \approx u(t_n)$ for $1 \leq n \leq N$ to the model problem (1.1), starting from an approximation $U_0 \approx u_0$ is defined by:

$$U^n - U^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (A\vec{U})(t)dt = \int_{t_{n-1}}^{t_n} f(t)dt =: \tilde{f}^n. \tag{3.3}$$

Integrate and using the definition of $\vec{U}$, we find that

$$\begin{aligned}\int_{t_{n-1}}^{t_n} D^\alpha(A\vec{U})(t)dt &= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\omega_{1-\alpha}(t_n-s)-\omega_{1-\alpha}(t_{n-1}-s))A\vec{U}(s)ds \\ &\quad + \int_{t_{n-1}}^{t_n}\omega_{1-\alpha}(t_n-s)(A\vec{U})(s)ds \\ &= \sum_{j=1}^{n-1}\omega_{nj}AU^j + \omega_{nn}AU^n,\end{aligned}$$

where, for $1 \le j \le n-1$

$$\omega_{nj} = \int_{t_{j-1}}^{t_j}(\omega_{1-\alpha}(t_n-s)-\omega_{1-\alpha}(t_{n-1}-s))ds,$$

and

$$\omega_{nn} = \int_{t_{n-1}}^{t_n}\omega_{1-\alpha}(t_n-s)ds = \omega_{2-\alpha}(k_n).$$

Therefore, the approximate solution U^n can be computed as follows:

$$(I+\omega_{nn}A)U^n = U^{n-1} + \tilde{f^n} - \sum_{j=1}^{n-1}\omega_{nj}AU^j. \tag{3.4}$$

where I denotes the identity operator on $L_2(\Omega)$. Hence, at each step we must solve an elliptic problem, so the scheme is implicit as mentioned earlier.

3.2.2 Stability

In this section, we prove the stability of the numerical scheme.

Stability: Small change in the data produce a small change in the solution.

First we have to show $U^n \in H^2(\Omega) \cap H_0^1(\Omega)$. This can be done by induction hypothesis.

For $n = 1$, taking the inner product of (3.4) with the eigenfunction ϕ_m, then we have

$$\langle U^1 - U^0 + \omega_{11} A U^1, \phi_m \rangle = \langle \tilde{f}^1, \phi_m \rangle,$$

since

$$\langle U^1, \phi_m \rangle = U_m^1, \quad \langle U^0, \phi_m \rangle = U_m^0, \text{ and } \quad \langle \tilde{f}^1, \phi_m \rangle = \tilde{f}_m^1,$$

$$(1 + \omega_{11}\lambda_m) U_m^1 = \tilde{f}_m^1 + U_m^0$$

We have

$$U_m^1 = \frac{1}{(1 + \omega_{11}\lambda_m)} (\tilde{f}_m^1 + U_m^0).$$

Next we show, $U^1 \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned}
\|AU^1\|^2 &= \langle AU^1, AU^1 \rangle \\
&= \langle \sum_{m=0}^{\infty} \lambda_m U_m^1 \phi_m, \sum_{m=0}^{\infty} \lambda_m U_m^1 \phi_m \rangle \\
&= \sum_{m=0}^{\infty} (\lambda_m U_m^1)^2 \\
&= \sum_{m=0}^{\infty} (\frac{\lambda_m}{1 + \omega_{11}\lambda_m})^2 (\tilde{f}_m^1 + U_m^0)^2.
\end{aligned}$$

By using the inequality, $\left(\frac{\lambda_m}{1+\omega_{11}\lambda_m}\right)^2 \leq \frac{\lambda_m^2}{\lambda_m^2\omega_{11}^2} = \frac{1}{\omega_{11}^2}$.

we get

$$\|AU^1\|^2 \leq \frac{1}{\omega_{11}^2}\sum_{m=0}^{\infty}[\tilde{f}_m^1 + U_m^0]^2.$$

Therefore,

$$\|AU^1\| \leq \frac{1}{\omega_{11}}\|f^1 + U^0\|.$$

But f^1, $U^0 \in L_2(\Omega)$, so $AU^1 \in L_2(\Omega)$. Therefore, by the elliptic regularity property, $U^1 \in H^2(\Omega) \cap H_0^1(\Omega)$.

Assume that $U^{n-1} \in H^2(\Omega) \cap H_0^1(\Omega)$, then we want to show that $U^n \in H^2(\Omega) \cap H_0^1(\Omega)$. This can be done as follows from (3.4), the fourier coefficients of U^n are given by

$$U_m^n = \frac{1}{(1+\omega_{nn}\lambda_m)}\left\{U_m^{n-1} + \tilde{f}_m^n - \sum_{j=1}^{n-1}\omega_{nj}\lambda_m U_m^j\right\}.$$

So,

$$\begin{aligned}
\|AU^n\|^2 &= \langle AU^n, AU^n\rangle \\
&= \langle\sum_{m=0}^{\infty}\lambda_m U_m^n\phi_m, \sum_{m=0}^{\infty}\lambda_m U_m^n\phi_m\rangle \\
&\leq \frac{1}{\omega_{nn}^2}\sum_{m=0}^{\infty}[\tilde{f}_m^n + U_m^{n-1} - \sum_{j=1}^{n-1}\omega_{nj}\lambda_m U_m^j]^2.
\end{aligned}$$

Using $(a-b)^2 \leq 2a^2 + 2b^2$. Then we have

$$\|AU^n\|^2 \leq \frac{2}{\omega_{nn}^2}\sum_{m=0}^{\infty}\|f^n + U^{n-1}\|^2 + \frac{2}{\omega_{nn}^2}\sum_{m=0}^{\infty}\omega_{nj}^2\|AU^j\|^2.$$

This implies

$$\|AU^n\| \leq \frac{\sqrt{2}}{\omega_{nn}}\|f^n + U^{n-1}\| + \frac{\sqrt{2}}{\omega_{nn}}\sum_{j=1}^{n-1}\omega_{nj}\|AU^j\|.$$

Therefore, by the induction hypothesis and using again the elipitic reqularity property $U^n \in H^2(\Omega) \cap H_0^1(\Omega)$.

Next we will show the stability of the numerical solution. Taking the inner product of (3.3) with U^n. This gives

$$\langle U^n - U^{n-1}, U^n\rangle + \int_{t_{n-1}}^{t_n}\langle D^\alpha(A\vec{U})(t), \vec{U}(t)\rangle dt = \langle U^n, \tilde{f}^n\rangle.$$

The equation can be rearranged as follows

$$\|U^n\|^2 + \int_{t_n}^{t_{n-1}}\langle D^\alpha(A\vec{U})(t), U(t)\rangle dt = \langle U^n, \tilde{f}^n\rangle + \langle U^{n-1}, U^n\rangle.$$

Using the Cauchy-Schwartz inequality and $ab \leq \frac{(a+b)^2}{2}$,

$$\|U^n\|^2 - \|U^{n-1}\|^2 + 2\int_{t_n}^{t_{n-1}}\langle D^\alpha(A\vec{U}(t), \vec{U}(t)\rangle dt \leq 2\|U^n\|\|\tilde{f}^n\|.$$

Let $\|U^{n^*}\| = \max\limits_{0\leq n\leq N}\|U^n\|$. Summing the above equation from $n = 1$ to $n = n^*$ gives

$$\|U^{n^*}\|^2 - \|U^0\|^2 + 2\int_0^{t_{n^*}}\langle D^\alpha(A\vec{U})(t), \vec{U}(t)\rangle dt \leq 2\sum_{n=1}^{n^*}\|U^n\|\|\tilde{f}^n\|,$$

since by Theorem A.1 [16],

$$\int_0^{t_{n^*}} \langle D^\alpha(A\vec{U})(t), \vec{U}(t)\rangle dt \geq 0,$$

we have

$$\|U^{n^*}\|^2 \leq \|U^0\|^2 + 2\sum_{n=1}^{n^*} \|U^n\| \|\tilde{f}^n\|.$$

Thus

$$\|U^{n^*}\|^2 \leq \|U^0\| \|U^{n^*}\| + 2\|U^{n^*}\| \sum_{n=1}^{n^*} \|\tilde{f}^n\|.$$

Then we have,

$$\|U^{n^*}\| \leq \|U^0\| + 2\sum_{n=1}^{n^*} \|\tilde{f}^n\|.$$

Therefore, our numerical solution is stable.

3.2.3 Error Estimate

In this section we estimate the error $e^n = U^n - u(t_n)$ in the L_2 norm, where U^n and u are the solutions of (3.3) and (1.1) respectively.

Comparing (3.3) with (3.2), we observe that the error e^n satisfies

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha(A\vec{U} - Au)(t)dt = 0.$$

From definition, we have this $\vec{U} = \vec{e} + \vec{u}$, then inserting this in the above equation

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha(A\vec{e})(t)dt = \eta^{n-\frac{1}{2}}, \tag{3.5}$$

where

$$\eta^{n-\frac{1}{2}} = \int_{t_{n-1}}^{t_n} D^\alpha(Au - A\vec{u})(t)dt. \tag{3.6}$$

and since $e^0 = U^0 - u_0$, from stability implies that

$$\|e^n\| \leq \|U^0 - u_0\| + \sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\| \text{ for } 1 \leq n \leq N. \tag{3.7}$$

Replacing n by j in (3.6) and using the definition of D^α, then we have

$$\eta^{j-\frac{1}{2}} = \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)A(u(s) - u(t_i))ds.$$

Apply energy norm and summing from $j = 1$ to n, then we have

$$\begin{aligned}
\sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\| &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)\|A(u(s) - u(t_i))\|ds \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s) \int_{s}^{t_j} \|Au^{'}(q)\|dqds \\
&\leq k^{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|Au^{'}(q)\|dq \\
&\leq k^{1-\alpha} \sum_{j=1}^{n} t_j^{\sigma+\alpha-2} k_j.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\| \leq \begin{cases} Ck^{1-\alpha} t_1^{\sigma - \frac{1}{\gamma} + \frac{\alpha}{\sigma}} \leq Ck^{1-\alpha} k^{\gamma\sigma - 1 + \alpha} = Ck^{\gamma\sigma}, & \text{if } \gamma < \frac{1}{\sigma}, \\ Ck^{1-\alpha}(t_n^{\frac{\alpha}{\sigma}} - t_1^{\frac{\alpha}{\sigma}}) \leq Ck^{1-\alpha}(n^\alpha k^\alpha - k^\alpha) = Ck(n^\alpha - 1), & \text{if } \gamma = \frac{1}{\sigma}, \\ Ck^{1-\alpha} t_n^{\frac{\alpha}{\gamma}} t_n^{\sigma - \frac{1}{\gamma}} \leq Ck^{1-\alpha}(kn)^\alpha t_n^{\sigma - \frac{1}{\gamma}} = kn^\alpha t_n^{\sigma - \frac{1}{\gamma}}, & \text{if } \gamma > \frac{1}{\sigma}. \end{cases}$$

Then finally, we have

$$\|U^n - u(t_n)\| \leq \|U^0 - u(t_0)\| + C \times \begin{cases} k^{\gamma\sigma}, & \text{if } 1 \leq \gamma < \frac{1}{\sigma}, \\ k(n^\alpha - 1)\ , & \text{if } \gamma = \frac{1}{\sigma}, \\ k n^\alpha t_n^{\sigma - \frac{1}{\gamma}}, & \text{if } \gamma > \frac{1}{\sigma}. \end{cases}$$

3.3 Crank-Nicolson method

In this section the time stepping Crank-Nicolson scheme for the model problem (1.1) will be introduced. The proof of the stability and the convergence of this scheme will be discussed.

Crank-Nicolson scheme is the average of explicit scheme and the implicit scheme. We introduce the the function v defined on the time grid points by:

$$\bar{v}(t) = \frac{v^{n-1} + v^n}{2} \text{ for } \ t_{n-1} < t < t_n, v^n = v(t_n) \text{ and } \ 1 \leq n \leq N.$$

3.3.1 Scheme

To define Crank-Nicolson solution $U^n \approx u(t_n)$ for $1 \leq n \leq N$ to the model problem (1.1), starting from an approximation $U_0 \approx u_0$ is given by:

$$U^n - U^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{U})(t)dt = \int_{t_{n-1}}^{t_n} f(t)dt =: \tilde{f}^n. \tag{3.9}$$

Integrate and using the definition of $\bar{U}$, we find that

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} D^\alpha(A\bar{U})(t)dt \\
&= \int_0^{t_n} \omega_{1-\alpha}(t_n - s)A\bar{U}(s)ds - \int_0^{t_{n-1}} \omega_{1-\alpha}(t_{n-1} - s)A\bar{U}(s)ds \\
&= \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s)A\bar{U}(s)ds + \int_0^{t_{n-1}} (\omega_{1-\alpha}(t_{n-1} - s) - \omega_{1-\alpha}(t_{n-1} - s))A\bar{U}(s)ds \\
&= \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (\omega_{1-\alpha}(t_n - s) - \omega_{1-\alpha}(t_{n-1} - s))A\bar{U}(s)ds + \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s)A\bar{U}(s)ds \\
&= \frac{1}{2}\sum_{j=1}^{n-1} \omega_{nj}A(U^{j-1} + U^j) + \frac{1}{2}\omega_{nn}A(U^{n-1} + U^n),
\end{aligned}$$

where, for $1 \le j \le n-1$,

$$\omega_{nj} = \int_{t_{j-1}}^{t_j} (\omega_{1-\alpha}(t_n - s) - \omega_{1-\alpha}(t_{n-1} - s))ds,$$

and

$$\omega_{nn} = \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s)ds = \omega_{2-\alpha}(k_n).$$

Therefore, the approximate solution U^n can be computed as follows:

$$(2I + \omega_{nn}A)U^n = (2I - \omega_{nn})U^{n-1} + 2\tilde{f}^n - \sum_{j=1}^{n-1} \omega_{nj}A(U^{j-1} + U^j), \tag{3.10}$$

where I denotes the identity operator on $L_2(\Omega$. Hence, at each step we must solve an elliptic problem, so the scheme is implicit.

3.3.2 Stability

In this section, we will prove the stability of the numerical scheme. First we have to show $U^n \in H^2(\Omega) \cap H_0^1(\Omega)$ by assuming $U^0 \in H^2(\Omega) \cap H_0^1(\Omega)$. This can be done by induction hypothesis.

For $n = 1$, taking the inner product of (3.10) with the eigenfunction ϕ_m. Then we have

$$\langle 2U^1 - 2U^0 + \omega_{11} A(U^0 + U^1), \phi_m \rangle = \langle 2\tilde{f}^1, \phi_m \rangle,$$

since

$$\langle U^1, \quad \phi_m \rangle = U_m^1, \langle U^0, \phi_m \rangle = U_m^0, \text{ and } \quad \langle \tilde{f}^1, \phi_m \rangle = \tilde{f}_m^1,$$

$$(2 + \omega_{11}\lambda_m) U_m^1 = 2\tilde{f}_m^1 + 2U^0 - \omega_{11}\lambda_m U_m^0.$$

From this we have

$$U_m^1 = \frac{1}{(2 + \omega_{11}\lambda_m)} (2\tilde{f}_m^1 + (2 - \omega_{11}\lambda_m) U_m^0).$$

Next we show, $U^1 \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned} \|AU^1\|^2 &= \langle AU^1, AU^1 \rangle \\ &= \langle \sum_{m=0}^{\infty} \lambda_m U_m^1 \phi_m, \sum_{m=0}^{\infty} \lambda_m U_m^1 \phi_m \rangle \\ &= \sum_{m=0}^{\infty} (\lambda_m U_m^1)^2 \\ &= \sum_{m=0}^{\infty} (\frac{\lambda_m}{2 + \omega_{11}\lambda_m})^2 (2\tilde{f}_m^1 + (2 - \omega_{11}\lambda_m) U_m^0)^2. \end{aligned}$$

By using this inequality $\left(\frac{\lambda_m}{2+\omega_{11}\lambda_m}\right)^2 \leq \frac{\lambda_m^2}{\lambda_m^2\omega_{11}^2} = \frac{1}{\omega_{11}^2}$.

We get

$$\|AU^1\|^2 \leq \frac{1}{\omega_{11}^2}\sum_{m=0}^{\infty}[2(\tilde{f_m^1} + U_m^0) - \omega_{11}\lambda_m U_m^0]^2.$$

Using this $(a-b)^2 \leq 2(a+b)^2$, we have

$$\begin{aligned}
\|AU^1\|^2 &\leq \frac{1}{\omega_{11}^2}\sum_{m=0}^{\infty}2[[2(\tilde{f_m^1} + U_m^0)]^2 + [\omega_{11}\lambda_m)U_m^0]^2] \\
&\leq \frac{8}{\omega_{11}^2}\sum_{m=0}^{\infty}[f_m^1 + U_0^m]^2 + \frac{2}{\omega_{11}^2}\sum_{m=0}^{\infty}[\lambda_m U_0^m]^2 \\
&\leq \frac{8}{\omega_{11}^2}\|f^1 + U^0\|^2 + \frac{2}{\omega_{11}^2}\|AU^0\|^2.
\end{aligned}$$

Therefore,

$$\|AU^1\| \leq \frac{\sqrt{8}}{\omega_{11}}\|f^1 + U^0\| + \frac{\sqrt{2}}{\omega_{11}}\|AU^0\|.$$

But $f^1, AU^0 \in L_2(\Omega)$, so $AU^1 \in L_2(\Omega)$. Therefore, by the elliptic regularity property, $U^1 \in H^2(\Omega) \cap H_0^1(\Omega)$.

Assume that $U^{n-1} \in H^2(\Omega)\cap H_0^1(\Omega)$, then we want to show that $U^n \in H^2(\Omega)\cap H_0^1(\Omega)$. This can be done as follows from (3.10), the fourier coefficients of U^n are given by

$$U_m^n = \frac{1}{(2+\omega_{nn}\lambda_m)}\left\{(2-\omega_{nn}\lambda_m)U_m^{n-1} + 2\tilde{f_m^n} - \sum_{j=1}^{n-1}\omega_{nj}\lambda_m(U_m^{j-1} + U_m^j)\right\}.$$

So,

$$\begin{aligned}
\|AU^n\|^2 &= \langle AU^n, AU^n\rangle \\
&= \langle\sum_{m=0}^{\infty}\lambda_m U_m^n\phi_m, \sum_{m=0}^{\infty}\lambda_m U_m^n\phi_m\rangle \\
&\leq \frac{1}{\omega_{nn}}\sum_{m=0}^{\infty}[2\tilde{f}_m^n + 2U_m^{n-1} - \lambda_m\omega_{nn}U_m^{n-1} - \sum_{j=1}^{n-1}\omega_{nj}\lambda_m(U_m^{j-1}+U_m^j)]^2,
\end{aligned}$$

Using this $(a-(b+c)^2)^2 \leq 2(a^2+2(b^2+c^2))$.

Then we have

$$\|AU^n\|^2 \leq \frac{8}{\omega_{nn}^2}\|f^n + U^{n-1}\|^2 + 2\omega_{nn}^2|AU^{n-1}\| + \frac{8}{\omega_{nn}^2}\sum_{j=1}^{n-1}\omega_{nj}^2[\|AU^{j-1}\|^2 + \|AU^j\|^2].$$

This implies

$$\|AU^n\| \leq \frac{\sqrt{8}}{\omega_{nn}}\|f^n + U^{n-1}\| + \sqrt{2}\omega_{nn}\|AU^{n-1}\| + \frac{\sqrt{8}}{\omega_{nn}}\sum_{j=1}^{n-1}\omega_{nj}(\|AU^{j-1}\| + \|AU^j\|).$$

Therefore, by the induction hypothesis and using again the elipitic reqularity property $U^n \in H^2(\Omega)\cap H_0^1(\Omega)$.

Next we will show the stability of the numerical solution. Taking the inner product of (3.9) with $U^n + U^{n-1}$. This gives

$$\langle U^n - U^{n-1}, U^n + U^{n-1}\rangle + 2\int_{t_{n-1}}^{t_n}\langle D^\alpha(A\bar{U})(t), \bar{U}(t)\rangle dt = \langle U^n + U^{n-1}, \tilde{f}^n\rangle.$$

This equation can be rearranged to give

$$\|U^n\|^2 - \|U^{n-1}\|^2 + 2\int_{t_n}^{t_{n-1}} \langle D^\alpha(A\bar{U}(t), \bar{U}(t)\rangle dt = \langle U^n + U^{n-1}, \tilde{f}^n\rangle.$$

Let $\|U^{n^*}\| = \max_{0\le n\le N} \|U^n\|$. Summing the above equation from $n = 1$ to $n = n^*$ gives

$$\|U^{n^*}\|^2 - \|U^0\|^2 + 2\int_0^{t_{n^*}} \langle D^\alpha(A\bar{U})(t), \bar{U}(t)\rangle dt = \sum_{n=1}^{n^*} \langle U^n + U^{n-1}, \tilde{f}^n\rangle,$$

since by Theorem A.1 [16],

$$\int_0^{t_{n^*}} \langle D^\alpha(A\bar{U})(t), \bar{U}(t)\rangle dt \geq 0,$$

we have

$$\|U^{n^*}\|^2 \leq \|U^0\|^2 + \sum_{n=1}^{n^*} \langle U^n + U^{n-1} \tilde{f}^n\rangle.$$

Thus

$$\|U^{n^*}\|^2 \leq \|U^0\|\|U^{n^*}\| + 2\|U^{n^*}\| \sum_{n=1}^{n^*} \|\tilde{f}^n\|.$$

Then we have,

$$\|U^{n^*}\| \leq \|U^0\| + 2\sum_{n=1}^{n^*} \|\tilde{f}^n\|.$$

Therefore, our numerical solution is stable.

3.3.3 Error estimate

In this section we estimate the error $e^n = U^n - u(t_n)$ in the L_2 norm, where U^n and u are the solutions of (3.9) and (1.1) respectively.

Comparing (3.9) with (3.2), we observe that the error e^n satisfies

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{U} - u)(t)dt = 0.$$

From definition, we have this $\bar{U} = \bar{e} + \bar{u}$, then inserting this in the above equation

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha A(\bar{e})(t)dt = \eta^{n-\frac{1}{2}}, \tag{3.11}$$

where

$$\eta^{n-\frac{1}{2}} = \int_{t_{n-1}}^{t_n} D^\alpha A(u - \bar{u})dt. \tag{3.12}$$

and since $e^0 = U^0 - u_0$, from stability implies that

$$\|e^n\| \leq \|U^0 - u_0\| + 2\sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\| \text{ for } 1 \leq n \leq N. \tag{3.13}$$

By writing

$$\begin{aligned} u(t) - \bar{u}(t) =& \frac{u(t)}{2} - \frac{u(t_n)}{2} + \frac{u(t)}{2} - \frac{u(t_{n-1})}{2} + (t - t_{n-\frac{1}{2}})u^{'}(t_n) \\ &+ (t_{n-1} - t)\frac{u^{'}(t_n)}{2} + (t_n - t)\frac{u^{'}(t_n)}{2}. \end{aligned}$$

After we rearranged, we have

$$u(t) - \bar{u}(t) = \frac{1}{2}\int_{t}^{t_n}\int_{s}^{t_n} u^{''}(q)dq - \frac{1}{2}\int_{t_{n-1}}^{t}\int_{s}^{t_n} u^{''}(q)dq + (t - t_{n-\frac{1}{2}})u^{'}(t_n).$$

Let

$$u(t) - \bar{u}(t) = \hat{e_1}(t) + \hat{e_2}(t),$$

where

$$\hat{e_1}(t) = \frac{1}{2}\int\limits_{t}^{t_n}\int\limits_{s}^{t_n} u^{''}(q)dq - \frac{1}{2}\int\limits_{t_{n-1}}^{t}\int\limits_{s}^{t_n} u^{''}(q)dq,$$

and

$$\hat{e_2}(t) = (t - t_{n-\frac{1}{2}})u^{'}(t_n).$$

Then we can write $\eta^{j-\frac{1}{2}}$ as

$$\begin{aligned}
\eta^{j-\frac{1}{2}} &= \int_{t_{j-1}}^{t_j} D^{\alpha}(Au - A\bar{u}(t))dt \\
&= \int_{t_{n-1}}^{t_n} D^{\alpha}(A\hat{e_1} + A\hat{e_2})(t)dt \\
&= \int_{t_{n-1}}^{t_n} D^{\alpha}(A\hat{e_1})(t)dt + \int_{t_{n-1}}^{t_n} D^{\alpha}(A\hat{e_2})(t)dt.
\end{aligned}$$

Let

$$\eta_1^{j-\frac{1}{2}} = \int\limits_{t_{j-1}}^{t_j} D^{\alpha}(A\hat{e_1})(t)dt,$$

and

$$\eta_2^{j-\frac{1}{2}} = \int_{t_{j-1}}^{t_j} D^{\alpha}(A\hat{e_2})(t)dt.$$

Then,

$$\sum_{j=1}^{n}\|\eta^{j-\frac{1}{2}}\|=\sum_{j=1}^{n}\|\eta_1^{j-\frac{1}{2}}\|+\sum_{j=1}^{n}\|\eta_2^{j-\frac{1}{2}}\|,$$

we shall proceed to estimate the terms $\sum_{j=1}^{n}\|\eta_1^{j-\frac{1}{2}}\|$ and $\sum_{j=1}^{n}\|\eta_2^{j-\frac{1}{2}}\|$.

Let us first bound $A\hat{e_1}(s)$. By using the definition of $\hat{e_1}(t)$, we have

$$A\hat{e_1}(s)=\frac{1}{2}\int_s^{t_j}\int_s^{t_j}Au^{''}(q)dqdt-\frac{1}{2}\int_{t_{n-1}}^{t}\int_s^{t_j}Au^{''}(q)\,dqdt.$$

After applying energy norm, we have

$$\begin{aligned}
\|A\hat{e_1}(s)\| &\leq \frac{1}{2}\int_s^{t_j}\int_s^{t_j}\|Au^{''}(q))\|dqdt+\frac{1}{2}\int_{t_{j-1}}^{t}\int_s^{t_j}\|Au^{''}(q)\|dqdt\\
&\leq \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_s^{t_j}\|Au^{''}(q))\|dqdt+\frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t_j}\|Au^{''}(q))\|dqdt\\
&=\frac{1}{2}k_j\int_{j-1}^{t_j}\|Au^{''}(q))\|dq+\frac{1}{2}k_j\int_{t_{j-1}}^{t_j}\|Au^{''}(q))\|dq\\
&=k_j\int_{j-1}^{t_j}\|Au^{''}(q)\|dq.
\end{aligned}$$

Therefore,

$$\|A\hat{e_1}(s)\|\leq k_j\int_{j-1}^{t_j}\|Au^{''}(q)\|dq \text{ for } t_{j-1}<s<t_j.$$

Next we will bound the first term $\sum_{j=1}^{n}\|\eta_1^{j-\frac{1}{2}}\|$.

For $j=1$, we have

$$\eta_1^{\frac{1}{2}}=\int_0^{t_1}\omega_{1-\alpha}(t_1-s)A\hat{e_1}(s)ds.$$

Then, by applying energy norm we have

$$\|\eta_1^{\frac{1}{2}}\| \leq k_1|\omega_{2-\alpha}(k_1)| \int_0^{t_1} \|A(u^{''}(q))\| dq,$$

since

$$|\omega_{2-\alpha}(k_1)| \leq k_1^{1-\alpha}.$$

Therefore,

$$\|\eta_1^{\frac{1}{2}}\| \leq k_1^{2-\alpha} \int_0^{t_1} \|A(u^{''}(q))\| dq. \tag{3.14}$$

For $j \geq 2$,

$$\begin{aligned}\eta_1^{j-\frac{1}{2}} = & \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} [\omega_{1-\alpha}(t_j - s) - \omega_{1-\alpha}(t_{j-1} - s)] A\hat{e_1}(s) ds \\ & + \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s) A\hat{e_1}(s) ds,\end{aligned}$$

then

$$\begin{aligned}\|\eta_1^{j-\frac{1}{2}}\| \leq & \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} [\omega_{1-\alpha}(t_j - s) - \omega_{1-\alpha}(t_{j-1} - s)] \|A\hat{e_1}(s)\| ds \\ & + \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s) \|A\hat{e_1}(s)\| ds,\end{aligned}$$

and summing from $j = 2$ to $j = n$ gives

$$\sum_{j=2}^{n}\|\eta_1^{j-\frac{1}{2}}\|\leq\sum_{j=2}^{n}\sum_{i=1}^{j-1}\int_{t_{i-1}}^{t_i}[\omega_{1-\alpha}(t_j-s)-\omega_{1-\alpha}(t_{j-1}-s)]\|A\hat{e_1}(s)\|ds$$
$$+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\omega_{1-\alpha}(t_j-s)\|A\hat{e_1}(s)\|ds.$$

By changing the order of summation, we get

$$\begin{aligned}
\sum_{j=2}^{n}\|\eta_1^{j-\frac{1}{2}}\|\leq&\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int_{t_{i-1}}^{t_i}[\omega_{1-\alpha}(t_{j-1}-s)-\omega_{1-\alpha}(t_j-s)]\|A\hat{e_1}(s)\|ds\\
&+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\omega_{1-\alpha}(t_j-s)\|A\hat{e_1}(s)\|ds\\
=&\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}[\omega_{1-\alpha}(t_1-s)-\omega_{1-\alpha}(t_n-s)]\|A\hat{e_1}(s)\|ds\\
&+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\omega_{1-\alpha}(t_j-s)\|A\hat{e_1}(s)\|ds\\
=&\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\omega_{1-\alpha}(t_1-s)\|A\hat{e_1}(s)\|ds-\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\omega_{1-\alpha}(t_n-s)\|A\hat{e_1}(s)\|ds\\
&+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\omega_{1-\alpha}(t_j-s)\|A\hat{e_1}(s)\|ds\\
=&\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\omega_{1-\alpha}(t_1-s)\|A\hat{e_1}(s)\|ds-\int_{t_0}^{t_{n-1}}\omega_{1-\alpha}(t_n-s)\|A\hat{e_1}(s)\|ds\\
&+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\omega_{1-\alpha}(t_j-s)\|A\hat{e_1}(s)\|ds.
\end{aligned}$$

Using $\int\limits_{t_{n-1}}^{t_n} - \int\limits_{0}^{t_n} = - \int\limits_{0}^{t_{n-1}}$, we have

$$\begin{aligned}
\sum_{j=2}^{n} \|\eta_1^{j-\frac{1}{2}}\| \leq & \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_1 - s)\|A\hat{e_1}(s)\| ds + \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s)\|A\hat{e_1}(s)\| ds \\
& - \int_{t_0}^{t_n} \omega_{1-\alpha}(t_n - s)\|A\hat{e_1}(s)\| ds + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)\|A\hat{e_1}(s)\| ds \\
\leq & \int_{0}^{t_1} \omega_{1-\alpha}(t_1 - s)\|A\hat{e_1}(s)\| ds + \sum_{i=2}^{n-1} \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_1 - s)\|A\hat{e_1}(s)\| ds \\
& + \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s)\|A\hat{e_1}(s)\| ds - \int_{t_0}^{t_n} \omega_{1-\alpha}(t_n - s)\|A\hat{e_1}(s)\| ds \\
& + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)\|A\hat{e_1}(s)\| ds \\
= & \int_{0}^{t_1} \omega_{1-\alpha}(t_1 - s)\|A\hat{e_1}(s)\| ds - \int_{t_0}^{t_n} \omega_{1-\alpha}(t_n - s)\|A\hat{e_1}(s)\| ds \\
& + 2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)\|A\hat{e_1}(s)\| ds.
\end{aligned}$$

After rearranged we have,

$$\sum_{j=2}^{n} \|\eta_1^{j-\frac{1}{2}}\| \leq \int_{0}^{t_1} \omega_{1-\alpha}(t_1 - s)\|A\hat{e_1}(s)\| ds + 2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s)\|A\hat{e_1}(s)\| ds,$$

since

$$|\omega_{2-\alpha}(k_1)| \leq k_1^{1-\alpha}.$$

Then, we have

$$\sum_{j=2}^{n} \|\eta_1^{j-\frac{1}{2}}\| \leq k_1^{2-\alpha} \int_{0}^{t_1} \|A\hat{e_1}(s)\| ds + 2\sum_{j=2}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} \|A\hat{e_1}(s)\| ds. \tag{3.15}$$

From (3.14) and (3.15). We have

$$\begin{aligned}\sum_{j=1}^{n}\|\eta_1^{j-\frac{1}{2}}\| &\leq \|\eta^{\frac{1}{2}}\|+\sum_{j=2}^{n}\|\eta_1^{n-\frac{1}{2}}\| \\ &= 2\sum_{j=1}^{n} k_j^{2-\alpha}\int_{t_{j-1}}^{t_j}\|A\hat{e_1}(s)\|ds.\end{aligned} \tag{3.16}$$

Next we will define definition and lemmas. That will help us to estimate the second term.

(1) For $1 \leq i \leq j \leq j \leq N$,

$$K_\alpha^{i,j} = \frac{-\alpha}{2\Gamma(1+\alpha)}\int\limits_{t_{i-1}}^{t_i}(s-t_{i-1})(t_i-s)(t_j-s)^{\alpha-1}.$$

By applying integration by parts. We have,

$$K_\alpha^{i,j} = \frac{-\alpha}{2\Gamma(1-\alpha)}\int\limits_{t_{i-1}}^{t_i}(s-t_{j-1})(t_j-s)^{\alpha} = -\frac{\alpha}{2}\omega_{3-\alpha}(k_j).$$

(2) Since,

$$k_j \geq k_{j-1},$$

then we have

$$K_\alpha^{j,i} - K_\alpha^{j-1,i-1} = -\frac{\alpha}{2}(\omega_{3+\alpha}(k_j) - \omega_{3+\alpha}(k_{j-1}) \geq 0.$$

Lemma 3.2 For $2 \le i < j$, we have

$$k_i^3 K_\alpha^{i-1,j-1} \ge k_{i-1}^3 K_\alpha^{i,j}.$$

Lemma 3.3 For $1 \le i < j \le n$ with $n \le N$ we have

$$\sum_{j=i+1}^{n} K_\alpha^{i,j} \le \frac{k_i}{2} \omega_{2-\alpha}(k_i).$$

Next we will estimate the second term $\sum_{j=1}^{n} \|\eta_2^{j-\frac{1}{2}}\|$

$$\begin{aligned}
\eta_2^{j-\frac{1}{2}} =& \int_{t_{j-1}}^{t_j} D^\alpha (A\hat{e_2})(t)dt \\
=& \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} [\omega_{1-\alpha}(t_j - s) - \omega_{1-\alpha}(t_{j-1} - s)] A\hat{e_1}(s)ds \\
&+ \int_{t_{j-1}}^{t_j} \omega_{1-\alpha}(t_j - s) A\hat{e_1}(s)ds \\
=& \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_j - s) A\hat{e_2}(s)ds \\
&- \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} [\omega_{1-\alpha}(t_{j-1} - s) A\hat{e_2}(s)ds
\end{aligned}.$$

Then

$$\begin{aligned}
\eta_2^{j-\frac{1}{2}} =& \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_j - s)(s - t_{i-\frac{1}{2}}) Au'(t_i)ds \\
&- \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_{j-1} - s)(s - t_{i-\frac{1}{2}}) Au'(t_i)ds
\end{aligned}$$

$$
\begin{aligned}
\eta_2^{j-\frac{1}{2}} =& \sum_{i=1}^{j} Au^{'}(t_i) \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_j - s)(s - t_{i-\frac{1}{2}})ds \\
&- \sum_{i=1}^{j-1} Au^{'}(t_i) \int_{t_{i-1}}^{t_i} \omega_{1-\alpha}(t_{j-1} - s)(s - t_{i-\frac{1}{2}})ds.
\end{aligned}
$$

By using (1)

$$
\begin{aligned}
\eta_2^{j-\frac{1}{2}} =& \sum_{i=1}^{j} Au^{'}(t_i)K_\alpha^{i,j} - \sum_{i=1}^{j-1} Au^{'}(t_i)K_\alpha^{i,j-1} \\
=& Au^{'}(t_j)K_\alpha^{i,j} - Au^{'}(t_j)K_\alpha^{j-1,j-1} + \sum_{i=1}^{j-1} Au^{'}(t_i)K_\alpha^{i,j} \\
&- \sum_{i=1}^{j-2} Au^{'}(t_i)K_\alpha^{i,j-1}.
\end{aligned}
$$

To proceed in our proof, we introduce the following notations

$$
\eta_{21}^{j-\frac{1}{2}} = Au^{'}(t_j)K_\alpha^{i,j} - Au^{'}(t_j)K_\alpha^{j-1,j-1},
$$

and

$$
\eta_{22}^{j-\frac{1}{2}} = \sum_{i=1}^{j-1} Au^{'}(t_i)K_\alpha^{i,j} - \sum_{i=1}^{j-2} Au^{'}(t_i)K_\alpha^{i,j-1}.
$$

Then

$$
\sum_{j=1}^{n} \|\eta_2^{j-\frac{1}{2}}\| \leq \sum_{j=1}^{n} \|\eta_{21}^{j-\frac{1}{2}}\| + \sum_{j=1}^{n} \|\eta_{22}^{j-\frac{1}{2}}\|.
$$

Next we will bound $\sum_{j=1}^{n} \|\eta_{21}^{j-\frac{1}{2}}\|$. For $j = 1$, we have

$$
\eta_{21}^{\frac{1}{2}} = \int_0^{t_1} \omega_{1-\alpha}(t_1 - s)A\hat{e_1}(s)ds = Au^{'}(t_1)K_\alpha^{1,1}.
$$

Then, we have

$$\|\eta_{21}^{\frac{1}{2}}\| \leq \|Au^{'}(t_1)K_{\alpha}^{1,1}\|.$$

For $j \geq 2$, we have

$$\begin{aligned}\eta_{21}^{j-\frac{1}{2}} &= Au^{'}(t_j)K_{\alpha}^{i,j} - Au^{'}(t_j)K_{\alpha}^{j-1,j-1}\\ &= (Au^{'}(t_j) - Au^{'}(t_{j-1}))K_{\alpha}^{i,j} + Au^{'}(t_j)(K_{\alpha}^{i,j} - K_{\alpha}^{j-1,j-1}),\end{aligned}$$

and so

$$\|\eta_{21}^{j-\frac{1}{2}}\| \leq \|Au^{'}(t_j) - Au^{'}(t_{j-1})\|K_{\alpha}^{i,j} + \|Au^{'}(t_j)\|(K_{\alpha}^{i,j} - K_{\alpha}^{j-1,j-1}).$$

By writing

$$Au^{'}(t_{j-1})K_{\alpha}^{i,j} = (Au^{'}(t_{j-1}) - Au^{'}(t_j))K_{\alpha}^{i,j} + Au^{'}(t_j)K_{\alpha}^{i,j}.$$

We obtain

$$\sum_{j=2}^{n}\|\eta_{21}^{j-\frac{1}{2}}\| \leq 2\sum_{j=2}^{n}\|(Au^{'}(t_j) - Au^{'}(t_{j-1}))K_{\alpha}^{i,j}\| + \|Au^{'}(t_j)_{\alpha}^{i,j}\| + \|Au^{'}(t_1)_{\alpha}^{1,1}\|.$$

Therefore,

$$\sum_{j=1}^{n}\|\eta_{21}^{j-\frac{1}{2}}\| \leq \|\eta_{21}^{\frac{1}{2}}\| + \sum_{j=2}^{n}\|\eta_{21}^{j-\frac{1}{2}}\|,$$

$$\begin{aligned}\sum_{j=1}^{n}\|\eta_{21}^{j-\frac{1}{2}}\| &\leq 2\sum_{j=2}^{n}\|(Au^{'}(t_j)-Au^{'}(t_{j-1}))K_\alpha^{i,j}\|+\|Au^{'}(t_n)\|K_\alpha^{n,n}\\ &\leq 2\sum_{j=2}^{n}K_\alpha^{i,j}\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|\,dq+\|Au^{'}(t_n)\|K_\alpha^{n,n}\\ &\leq 2\sum_{j=2}^{n}\frac{\alpha}{2}\omega_{3-\alpha}(k_j)\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|\,dq+\frac{\alpha}{2}\|Au^{'}(t_n)\|\omega_{3-\alpha}(k_n)\end{aligned}$$

$$\sum_{j=1}^{n}\|\eta_{21}^{j-\frac{1}{2}}\|\leq \alpha(\sum_{j=2}^{n}(k_j^{2-\alpha}\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|\,dq+\frac{1}{2}\|Au^{'}(t_n)\|\omega_{3-\alpha}(k_n)). \qquad (3.17)$$

Next we will bound $\sum_{j=1}^{n}\|\eta_{22}^{j-\frac{1}{2}}\|$,

$$\eta_{22}^{j-\frac{1}{2}}=\sum_{i=1}^{j-1}Au^{'}(t_i)K_\alpha^{i,j}-\sum_{i=1}^{j-2}Au^{'}(t_i)K_\alpha^{i,j-1}.$$

For $j=1$, $\eta_{22}^{j-\frac{1}{2}}=0$, while $\eta_{22}^{j-\frac{1}{2}}=Au^{'}(t_1)K_\alpha^{1,1}$ for $j=2$.

For $j\geq 3$,

$$\begin{aligned}\eta_{22}^{j-\frac{1}{2}} &=\sum_{i=1}^{j-1}Au^{'}(t_i)K_\alpha^{i,j}-\sum_{i=1}^{j-2}Au^{'}(t_i)K_\alpha^{i,j-1}\\ &=\sum_{i=1}^{j-1}Au^{'}(t_i)K_\alpha^{i,j}-\sum_{i=2}^{j-1}Au^{'}(t_{i-1})K_\alpha^{i-1,j-1}\end{aligned}$$

Let

$$\eta_{221}^{j-\frac{1}{2}}=Au^{'}(t_1)K_\alpha^{1,1}+\sum_{i=2}^{j-1}[Au^{'}(t_i)-Au^{'}(t_{i-1})\frac{k_{i-1}^3}{k_i^3}]K_\alpha^{i,j},$$

and

$$\eta_{222}^{j-\frac{1}{2}}=-\sum_{i=2}^{j-1}Au^{'}(t_{i-1})(K_\alpha^{i-1,j-1}-\frac{k_{i-1}^3}{k_i^3}K_\alpha^{i,j}).$$

Then,

$$\eta_{22}^{j-\frac{1}{2}} = \eta_{221}^{j-\frac{1}{2}} + \eta_{222}^{j-\frac{1}{2}}.$$

Summing from $j = 1$ to $j = n$

$$\begin{aligned}\sum_{j=1}^{n} \|\eta_{22}^{j-\frac{1}{2}}\| &= \|\eta_{22}^{\frac{1}{2}}\| + \|\eta_{22}^{\frac{3}{2}}\| + \sum_{j=3}^{n} \|\eta_{22}^{j-\frac{1}{2}}\| \\ &= \|Au^{'}(t_1)\| K_{\alpha}^{1,1} + \sum_{j=3}^{n} \|\eta_{22}^{j-\frac{1}{2}}\| \\ &\leq \|Au^{'}(t_1)\| K_{\alpha}^{1,1} + \sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\| + \sum_{j=3}^{n} \|\eta_{222}^{j-\frac{1}{2}}\|.\end{aligned}$$

Next we will bound, $\sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\|$, and $\sum_{j=3}^{n} \|\eta_{222}^{j-\frac{1}{2}}\|$
First let us bound, $\sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\|$,

$$\sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\| \leq \sum_{j=3}^{n} \|Au^{'}(t_1)\| K_{\alpha}^{1,j} + \sum_{j=3}^{n} \sum_{i=2}^{j-1} \|Au^{'}(t_i) - Au^{'}(t_{i-1}) \frac{k_{i-1}^3}{k_i^3}\| K_{\alpha}^{i,j},$$

change the order of summation

$$\sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\| \leq \|Au^{'}(t_1)\| \sum_{j=3}^{n} K_{\alpha}^{1,j} + \sum_{i=3}^{n-1} \sum_{j=i+1}^{n} \|Au^{'}(t_i) - Au^{'}(t_{i-1}) \frac{k_{i-1}^3}{k_i^3}\| K_{\alpha}^{i,j}$$

$$\sum_{j=3}^{n} \|\eta_{221}^{j-\frac{1}{2}}\| \leq \|Au^{'}(t_1)\| \sum_{j=3}^{n} K_{\alpha}^{1,j} + \sum_{i=3}^{n-1} \|Au^{'}(t_i) - Au^{'}(t_{i-1}) \frac{k_{i-1}^3}{k_i^3}\| \sum_{j=i+1}^{n} K_{\alpha}^{i,j}. \quad (3.18)$$

Next we will bound, $\sum_{j=3}^{n}\|\eta_{222}^{j-\frac{1}{2}}\|$,

$$\eta_{222}^{j-\frac{1}{2}}=-\sum_{i=2}^{j-1}Au^{'}(t_{i-1})(K_{\alpha}^{i-1,j-1}-\frac{k_{i-1}^{3}}{k_{i}^{3}}K_{\alpha}^{i,j}).$$

Then, we have

$$\sum_{j=3}^{n}\|\eta_{222}^{j-\frac{1}{2}}\|\leq\sum_{j=3}^{n}\sum_{i=2}^{j-1}Au^{'}(t_{i-1})(K_{\alpha}^{i-1,j-1}-\frac{k_{i-1}^{3}}{k_{i}^{3}}K_{\alpha}^{i,j}).$$

By changing the order of summation, we get

$$\sum_{j=3}^{n-1}\|\eta_{222}^{j-\frac{1}{2}}\|\leq\sum_{i=2}^{n}\sum_{j=i+1}^{n}Au^{'}(t_{i-1})(K_{\alpha}^{i-1,j-1}-\frac{k_{i-1}^{3}}{k_{i}^{3}}K_{\alpha}^{i,j)},$$

and a shift indices gives

$$\begin{aligned}\sum_{j=3}^{n}\|\eta_{222}^{j-\frac{1}{2}}\|\leq&\sum_{i=1}^{n-2}\|Au^{'}(t_{i})\|\sum_{j=i+1}^{n-1}K_{\alpha}^{i,j}-\sum_{i=2}^{n-1}\|Au^{'}(t_{i})\|\frac{k_{i-1}^{3}}{k_{i}^{3}}\sum_{j=i+1}^{n}K_{\alpha}^{i,j}\\=&\|Au^{'}(t_{1})\|\sum_{j=2}^{n-1}K_{\alpha}^{1,j}+\sum_{i=2}^{n-2}\|Au^{'}(t_{i})\|\sum_{j=i+1}^{n-1}K_{\alpha}^{i,j}\\&-\sum_{i=2}^{n-1}\|Au^{'}(t_{i-1})\|\frac{k_{i-1}^{3}}{k_{i}^{3}}\sum_{j=i+1}^{n}K_{\alpha}^{i,j}\\\leq&\|Au^{'}(t_{1})\|\sum_{j=2}^{n-1}K_{\alpha}^{1,j}+\sum_{i=2}^{n-1}\|Au^{'}(t_{i})\\&-Au^{'}(t_{i-1})\|\frac{k_{i-1}^{3}}{k_{i}^{3}}\sum_{j=i+1}^{n}K_{\alpha}^{i,j}.\end{aligned}$$

Then, we have

$$\sum_{j=3}^{n}\|\eta_{222}^{j-\frac{1}{2}}\|\leq\|Au^{'}(t_1)\|\sum_{j=2}^{n-1}K_{\alpha}^{1,j}+\sum_{i=2}^{n}[\int_{t_{i-1}}^{t_i}\|Au^{''}\|\,dq \\ +3(k_i-k_{i-1})k_i^{-1}]\|Au^{'}(t_{i-1})\|\sum_{j=i+1}^{n}K_{\alpha}^{i,j}. \tag{3.19}$$

Finally, by combining equation (3.18) and (3.19), We have

$$\sum_{j=1}^{n}\eta_{22}^{j-\frac{1}{2}}\leq C(\|Au^{'}(t_1)\|k_1^{2-\alpha}+\sum_{j=2}^{n}k_j^{2-\alpha}\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|\,dq \\ +\sum_{j=2}^{n}k_j^{1-\alpha}(k_j-k_{j-1})\|Au^{''}(t_{j-1})\|\,dq). \tag{3.20}$$

Therefore, $\sum_{j=1}^{n}\|\eta_2^{j-\frac{1}{2}}\|\leq\sum_{j=1}^{n}\|\eta_{21}^{j-\frac{1}{2}}\|+\sum_{j=1}^{n}\|\eta_{22}^{j-\frac{1}{2}}\|$,

$$\sum_{j=1}^{n}\|\eta_2^{j-\frac{1}{2}}\|\leq C[k_1^{2-\alpha}\|Au^{'}(t_1)\|+k_n^{2-\alpha}\|Au^{'}(t_n)\| \\ +\sum_{j=1}^{n}k_j^{2-\alpha}\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|dq \\ +\sum_{j=2}^{n}k_j^{1-\alpha}(k_j-k_{j-1})\|Au^{''}(t_{j-1})\|dq].$$

Then, finally equation (3.13) becomes

$$\|U^n-u(t_n)\|\leq\|U^0-u(t_0)\|+k_1^{2-\alpha}\|Au^{'}(t_1)\|+k_n^{2-\alpha}\|Au^{'}(t_n)\| \\ +\sum_{j=1}^{n}k_j^{2-\alpha}\int_{t_{j-1}}^{t_j}\|Au^{''}(q)\|\,dq \\ +\sum_{j=2}^{n}k_j^{1-\alpha}(k_j-k_{j-1})\|Au^{''}(t_{j-1})\|\,dq. \tag{3.21}$$

Since $k_j^{1-\alpha} \geq k_{j-1}^{1-\alpha}$,

$$k_j^{1-\alpha}(k_j - k_{j-1})\|Au^{'}(t_{j-1})\| \leq (k_j^{]1-\alpha} - k_{j-1}^{1-\alpha})\|Au^{'}(t_{j-1})\|$$

$$\leq k_j^{2-\alpha}\|Au^{'}(t_j) - Au^{'}(t_{j-1})\| + k_j^{2-\alpha}\|Au^{'}(t_j)\| - k_{j-1}^{2-\alpha}\|Au^{'}(t_j)\|.$$

Then, we have

$$\sum_{j=2}^{n} k_j^{1-\alpha}(k_j - k_{j-1})\|Au^{''}(t_{j-1})\|\, dq \leq \sum_{j=2}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} \|Au^{'}(s)\|\, ds$$

$$+ k_n^{2-\alpha}\|Au^{'}(t_j)\| - k_1^{1-\alpha}\|Au^{'}(t_1)\|.$$

Substituting the above in equalities in (3.21) gives

$$\|U^n - u(t_n)\| \leq \|U^0 - u(t_0)\| + k_n^{2-\alpha}\|Au^{'}(t_n)\| + \sum_{j=1}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} \|Au^{''}(t)\|\, dt.$$

By using the regularity assumption (2.3) and Lemma (3.1), we will reduce the right hand side of inequality

$$k_n^{2-\alpha} t_n^{\sigma-2+\alpha} \leq (\gamma k t_n^{1-\frac{1}{\gamma}})^{2-\alpha} t_n^{\sigma-2+\alpha} = \gamma^{2-\alpha} k^{2-\alpha} t_n^{\sigma-\frac{(2+\alpha)}{\gamma}}.$$

Therefore,

$$k_n^{2-\alpha} t_n^{\sigma-2+\alpha} \leq \begin{cases} \gamma^{2-\alpha} k^{2-\alpha} (nk)^{(\sigma-\frac{(2+\alpha)}{\gamma})\gamma} = \gamma^{2-\alpha} n^{\gamma\sigma-(2+\alpha)} k^{\gamma\sigma} \leq C k^{\gamma\sigma}, & \text{if } \gamma < \frac{(2-\alpha)}{\sigma}, \\ C n^{\gamma\sigma} k^{\gamma\sigma} \leq C k^{2-\alpha}, & \text{if } \gamma \geq \frac{(2-\alpha)}{\sigma}, \end{cases}$$

and

$$\begin{aligned}\sum_{j=1}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} \|Au''(t)\| dt \leq & C \sum_{j=1}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} t^{\sigma+\alpha-3} dt \\ & \leq C \sum_{j=1}^{n} k_j^{2-\alpha} t_j^{\sigma+\alpha-3} k_j \\ & \leq C k^{2-\alpha} \int_{t_1}^{t_n} t^{\sigma-\frac{(2-\alpha)}{\gamma}-1} dt.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} k_j^{2-\alpha} \int_{t_{j-1}}^{t_j} \|Au''(t)\| dt \leq \begin{cases} Ck^{2-\alpha} t_1^{\sigma-\frac{(2-\alpha)}{\gamma}-1} = Ck^{\gamma\sigma}, & \text{if } \gamma < \frac{(2-\alpha)}{\sigma}, \\ Ck^{2-\alpha} \int_{t_1}^{t_n} t^{-1} dt = Ck^{2-\alpha} \log(\frac{t_n}{t_1}), & \text{if } \gamma = \frac{(2-\alpha)}{\sigma}, \\ Ck^{2-\alpha} t_n^{\sigma-\frac{2-\alpha}{\gamma}} \leq C(nk)^{\sigma\gamma} \leq Ck^{2-\alpha}, & \text{if } \gamma > \frac{(2-\alpha)}{\sigma}. \end{cases}$$

Then finally, we have

$$\|U^n - u(t_n)\| \leq \|U^0 - u(t_0)\| + C \times \begin{cases} k^{\gamma\sigma}, & \text{if } 1 \leq \gamma < \frac{(2-\alpha)}{\sigma}, \\ k^{2-\alpha} \log(\frac{t_n}{t_1}), & \text{if } \gamma = \frac{(2-\alpha)}{\sigma}, \\ k^{2-\alpha}, & \text{if } \gamma > \frac{(2-\alpha)}{\sigma}. \end{cases}$$

3.4 Numerical Results

In this section, we provide a purely time-dependent problem which enable us to illustrate the theoretical converge result from the time steeping schemes.

Example 3.1 *We consider*

$$\frac{du}{dt} + \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-s)u(s)ds = f(t) \text{ for } 0 < t < T \text{ with } u(0) = u_0.$$

Choosing the initial datum $u_0 = 1$ *and a source term* $f(t) = (2-\alpha)t^{1-\alpha}$. *This problem has exact solution*

$$u(t) = E_{1-\alpha}(-t^{1-\alpha}) + t\Gamma(3-\alpha)(1 - E_{1-\alpha,2}(-t^{1-\alpha})).$$

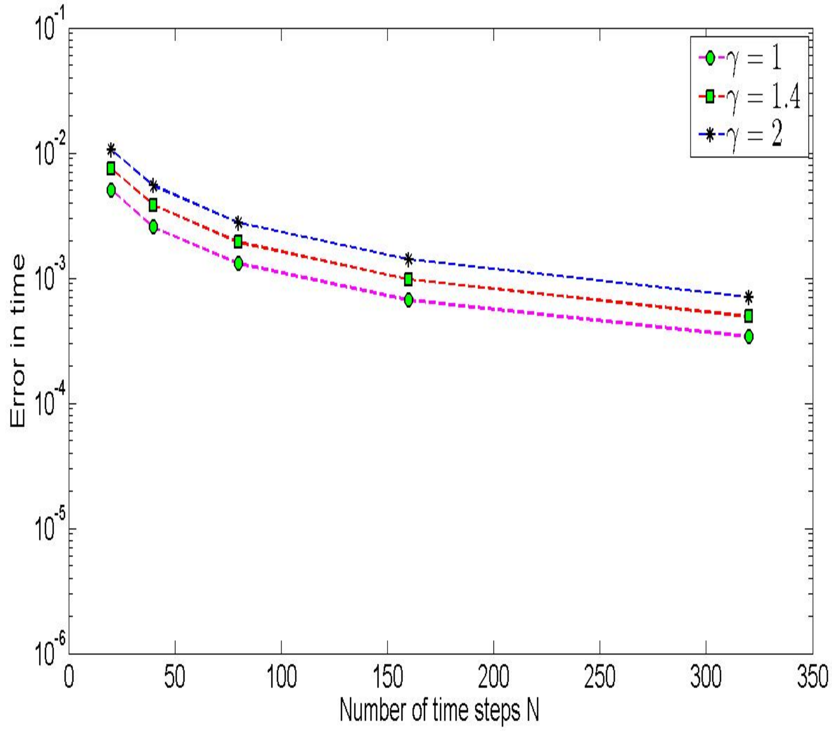
The regularity condition (2.3) holds for $\sigma = 2 - 2\alpha$ and next we will see our theoretical result with numerical result.

M.Error= Maximum nodal time error   OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 1.4$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	5.0582e-03	–	7.5226e-03	–	1.0623e-02	–
40	2.5451e-03	0.991	3.8327e-03	0.973	5.4594e-03	0.960
80	1.3109e-03	0.957	1.9390e-03	0.983	2.7757e-03	0.976
160	6.6857e-04	0.971	9.7657e-04	0.990	1.4020e-03	0.986
320	3.3804e-04	0.984	4.9044e-04	0.994	7.0528e-04	0.991
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 1.4$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	1.0294e-03	–	2.8946e-04	–	2.2880e-04	–
40	3.3795e-04	1.607	8.0470e-05	1.847	9.8960e-05	1.209
80	1.1119e-04	1.604	2.2630e-05	1.830	3.7070e-05	1.416
160	3.6630e-05	1.602	6.4100e-06	1.819	1.2790e-05	1.535
320	1.2080e-05	1.601	2.0900e-06	1.618	4.2100e-06	1.604

Table 3.1: Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.2$. We observe better orders for the CN scheme. The errors and convergence rates for the CN improved when the mesh is graded. We observe convergence rate of BW k^1 for $\gamma > \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.

Backward Euler



Crank-Nicolson

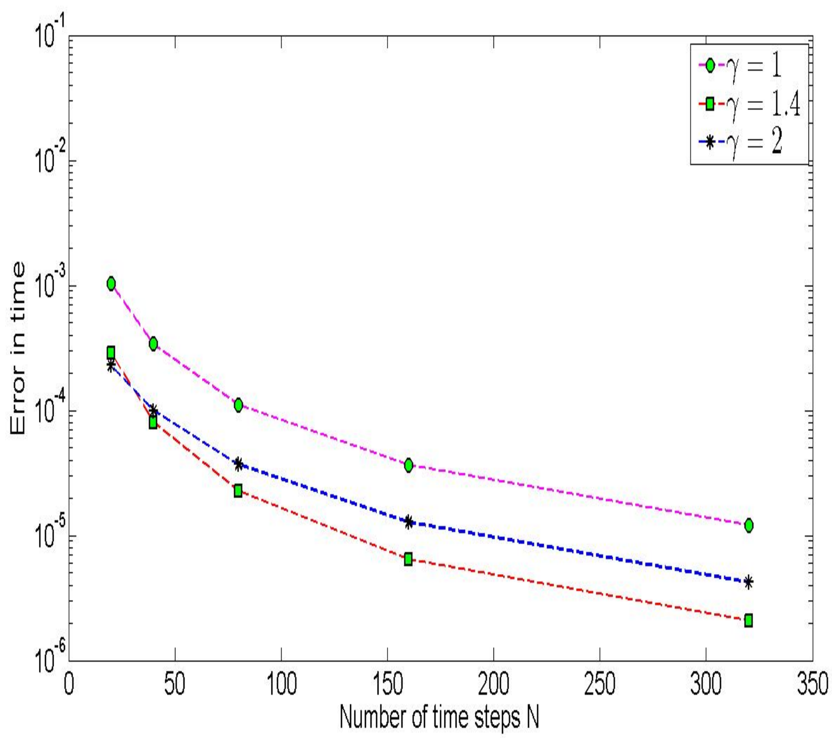


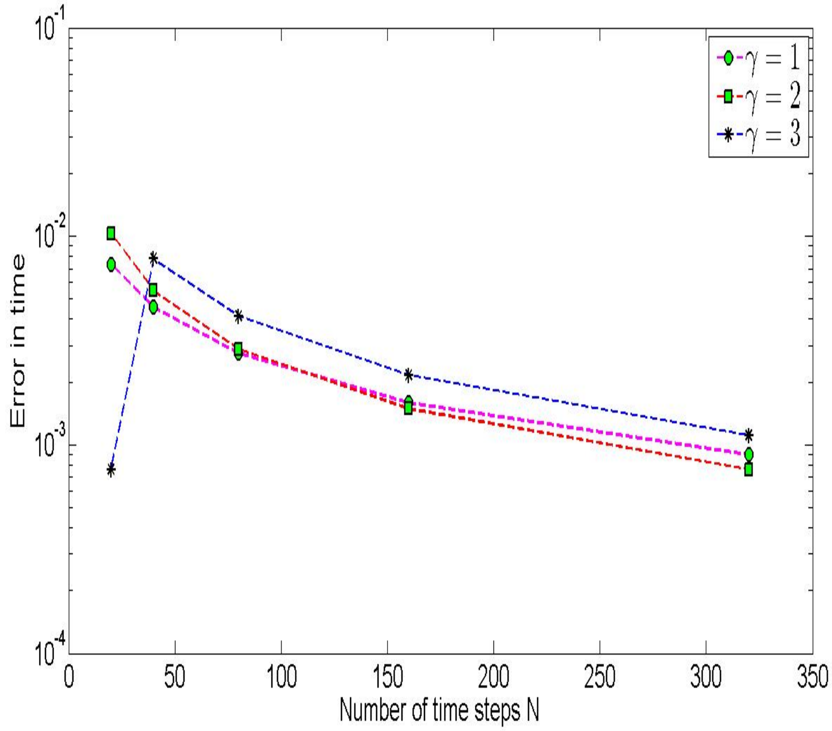
Figure 3.1: *Error against time steps for* $\alpha = 0.2$.

M.Error= Maximum nodal time error   OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	7.3558e-03	–	1.0284e-02	–	1.4345e-02	–
40	4.5649e-03	0.688	5.4984e-03	0.903	7.8127e-03	0.877
80	2.7517e-03	0.730	2.8811e-03	0.932	4.1462e-03	0.914
160	1.5911e-03	0.790	1.4885e-03	0.953	2.1611e-03	0.940
320	8.9386e-04	0.832	7.6154e-04	0.967	1.1124e-03	0.958
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	1.4214e-02	–	2.1347e-03	–	3.6948e-03	–
40	7.6443e-03	8.9500e-01	8.2073e-04	1.379	1.4612e-03	1.338
80	4.0219e-03	9.2600e-01	3.0649e-04	1.421	5.5484e-04	1.397
160	2.0843e-03	9.4800e-01	1.1240e-04	1.447	2.0560e-04	1.432
320	1.0688e-03	9.6300e-01	4.0740e-05	1.464	7.5020e-05	1.454

Table 3.2: *Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.5$. We observe better orders for the CN scheme. The errors and convergence rates for the CN improved when the mesh is graded. We observe convergence rate of order for BW k^1 for $\gamma > \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.*

Backward Euler



Crank-Nicolson

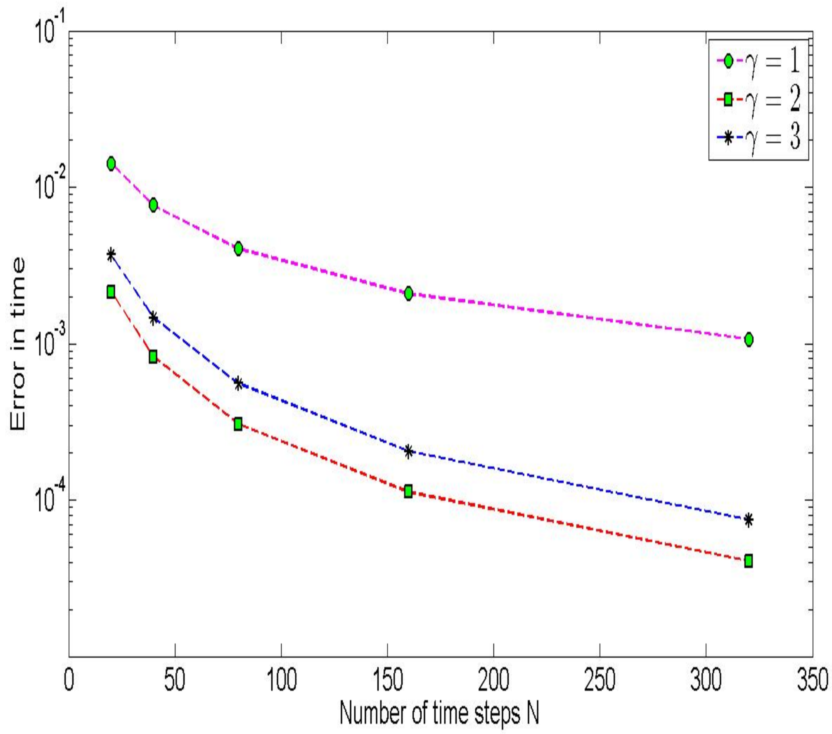


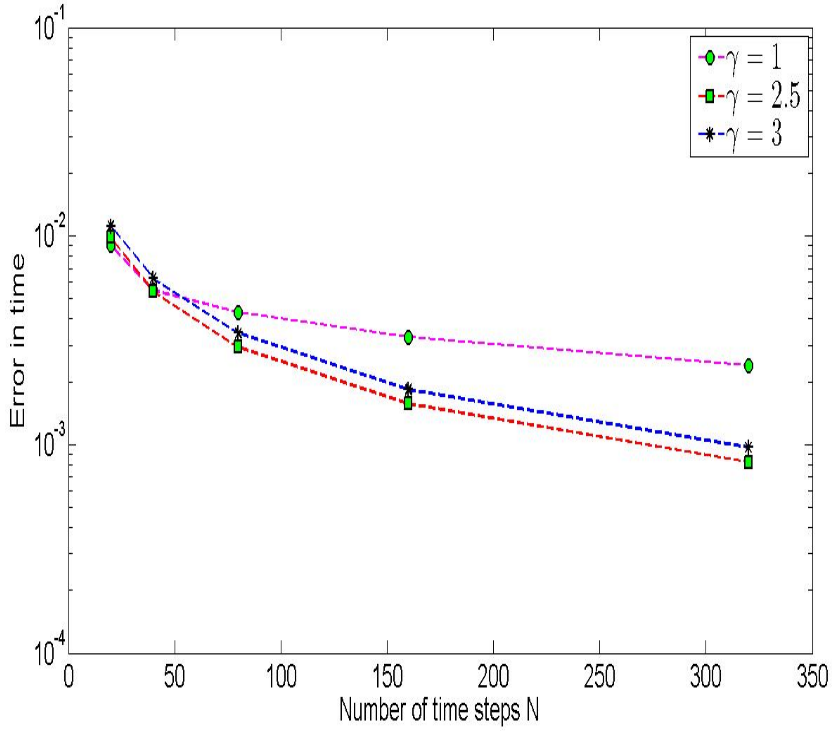
Figure 3.2: *Error against time steps for* $\alpha = 0.5$.

M.Error= Maximum nodal time error OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 2.5$		$\gamma = 3$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	9.0173e-03	–	9.8429e-03	–	1.1151e-02	–
40	5.4684e-03	0.722	5.4234e-03	0.860	6.2621e-03	0.832
80	4.3210e-03	0.340	2.9401e-03	0.883	3.4295e-03	0.869
160	3.2671e-03	0.403	1.5670e-03	0.908	1.8366e-03	0.901
320	2.3997e-03	0.445	8.2285e-04	0.929	9.6860e-04	0.923
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 2.5$		$\gamma = 3$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	4.8696e-02	–	5.9016e-03	–	7.3506e-03	–
40	3.5474e-02	0.457	2.5172e-03	1.229	3.1631e-03	1.216
80	2.5321e-02	0.486	1.0555e-03	1.254	1.3299e-03	1.25
160	1.7804e-02	0.508	4.3787e-04	1.269	5.5523e-04	1.26
320	1.2373e-02	0.525	1.8161e-04	1.270	2.3042e-04	1.267

Table 3.3: *Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.7$. We observe better orders for the CN scheme. The errors and convergence rates for the BW and CN improved when the mesh is graded. We observe convergence rate of order for BW k^1 for $\gamma > \frac{1}{\sigma}$ and $k^{\gamma\sigma}$ for $\gamma < \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.*

Backward Euler



Crank-Nicolson

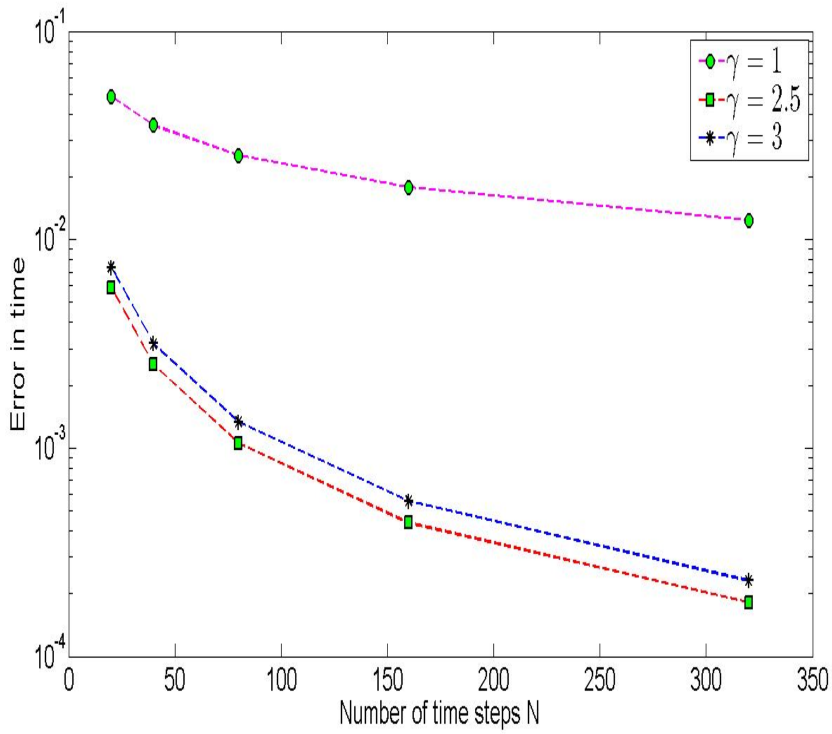


Figure 3.3: *Error against time steps for* $\alpha = 0.7$.

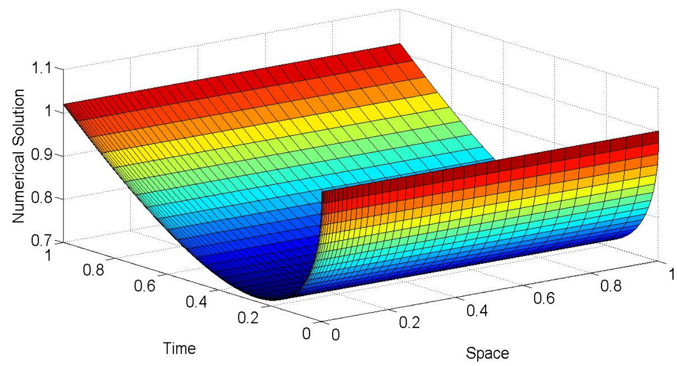


Figure 3.4: *Numerical solution for N=40,* $\gamma = 2$ *and* $\alpha = 0.5$.

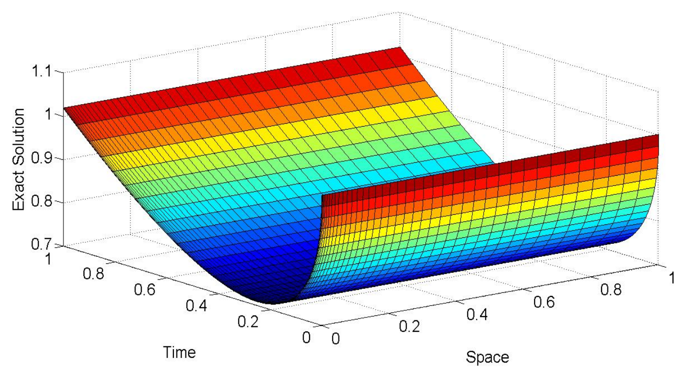


Figure 3.5: *Exact solution for a given problem.*

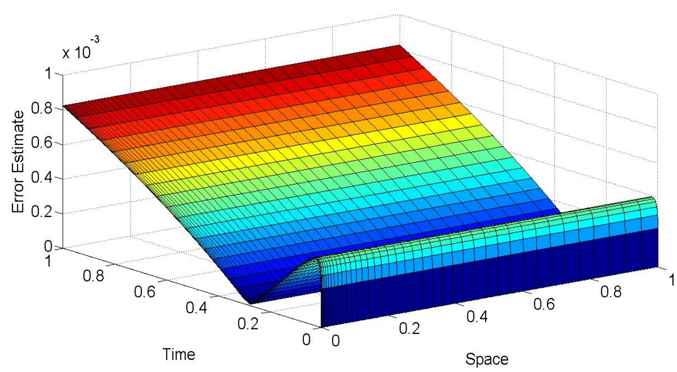


Figure 3.6: *Error estimate for N=40,* $\gamma = 2$ *and* $\alpha = 0.5$.

CHAPTER 4

SPATIAL DISCRETIZATION

In this chapter we semi-discretize the model problem (1.1) in space, using finite element method. Stability and convergence will be studied in details.

4.1 Finite Element Spaces

In practice, the spaces over which we solve variational problems associated with boundary value problems are called finite element spaces. We partition the given domain Ω into finitely many sub-domains and consider functions which reduce to a polynomial on each sub-domain. The sub-domains are called elements. There is no change conceptually in going from one dimension to two or three dimensions. The main practical difference is that instead of sub intervals in one dimension, the element become triangles or quadrilaterals in two dimensions, and tetrahedra, cubes, rectangular parallelepiped, etc. in three dimensions. For simplicity, we restrict our discussion primarily to one and two-dimensional cases. The idea of finite element method is to first split the computational domain into a set of elements K. The set of elements is known as the mesh and the vertices of the elements are known as the nodes. For an admissible triangulation (or subdivision) $\tau_h = \{K\}$ of Ω we require that $\bar{\Omega} = \cup_{K\in\tau_h} K$, and the intersection of any two distinct elements in τ_h either consists of a common face, common side or common vertex, or is empty. For any $K \in \tau_h$, let

$$h_k = diam(K), h = max_{K\in\tau_h} h_K,$$

and

$$\rho_K = sup\{diam(B) : \text{ B is a ball contained in K}\}.$$

We assume that τ_h is quasi-uniform, i.e we require the following conditions. The first condition is there exists a number $C > 0$, independent of h and K such that $\frac{\rho_K}{h_K} \geq C, \in \tau_h$ and the other condition is a parameter h approaches zero. These conditions mean that the triangles $K \in \tau_h$ are not allowed to be arbitrary thin, or equivalently, the angels of the triangles K are not allowed to be arbitrary small. The constant C is a measure of the smallest angle in any $K \in \tau_h$.
Next we introduce the finite element spaces V_h; this defined to be finite dimensional subspace, which consists of functions that polynomials of degree $\leq K$ in each element K , i.e.,

$$V_h := \{v \in H_0^1(\Omega) : v|_k \in \mathcal{P}_k(K), \forall K \in \tau_h \text{ and } v = 0 \text{ on } \Gamma\}.$$

4.2 Numerical solution in space

In this section we use continuous, piecewise-linear finite element over a quasi-uniform triangulation of Ω, so $u_h(t) : [0, T] \to V_h$ satisfies

$$\langle u_h^{'}(t), v\rangle + B(D^{\alpha}u_h(t), v(t)) = \langle f(t), v\rangle \forall v \in V_h \text{ and } 0 \leq t \leq T. \tag{4.1}$$

u_h can be written for each fixed t, of the basis function as follows:

$$u_h(x, t) = \sum_{i=1}^{M_h} \beta_i(t)\Phi_i(x) \ , \tag{4.2}$$

where $\{\Phi_i\}_{i=1}^{M_h}$ a basis for V_h where M_h is the total number of interior nodes. Insert this in (4.1) and choose $v = \Phi_j$ to find that this equation is equivalent to

$$\sum_{i=1}^{M_h} \beta_i^{'}(t)\langle \Phi_i, \Phi_j\rangle + \sum_{i=1}^{M_h} D^{\alpha}\beta_i(t)\langle a\nabla\Phi_i, \nabla\Phi_j\rangle = \langle f(t), \nabla\Phi_j\rangle, \tag{4.3}$$

where

$$\beta(t) = (\beta_i)_{M_h\times 1}, M = (\Phi_i, \Phi_j)_{M_h\times M_h} S = (a\nabla\Phi_i, \nabla\Phi_j)_{M_h\times M_h}, F(t) = (f(t), \Phi_j)_{M_h\times 1}.$$

Therefore, equation (4.3) becomes $M\beta^{'}(t) + SD^{\alpha}(\beta(t)) = F(t)$.

4.3 Stability

In this section, the stability of the FE solution will be shown. To do so, choose $v = u_h(t)$ and insert this in equation (4.1). Then we get

$$\langle u_h^{'}(t), u_h(t)\rangle + B(D^{\alpha}u_h(t), u_h(t)) = \langle f(t), u_h(t)\rangle.$$

But

$$\langle u_h^{'}(t), u_h(t)\rangle = \frac{1}{2}\frac{du}{dt}\|u_h(t)\|^2.$$

Thus by Cauchy-Schwartz inequality,

$$\frac{1}{2}\frac{du}{dt}\|u_h(t)\|^2 + B(D^{\alpha}u_h(t), u_h(t)) \leq \|f(t)\|\|u_h(t)\|. \tag{4.4}$$

Then integrate equation (4.4) from 0 to T yields

$$\|u_h(T)\|^2 + \int_0^T B(D^{\alpha}u_h(s), u_h(s))ds \leq \|u_h(0)\|^2 + 2\int_0^T \|f(s)\|\|u_h(s)\|ds.$$

Since by Theorem A.1 [16],

$$\int_0^T B(D^{\alpha}u_h(s), u_h(s))ds \geq 0,$$

we have

$$\|u_h(T)\|^2 \leq \|u_h(0)\|^2 + 2\int_0^T \|f(t)\|\|u_h(t)\|dt.$$

Setting

$$\|RHS(t)\| = \|u_h(0)\|^2 + 2\int_0^T \|f(s)\|\|u_h(s)\|ds.$$

Then, we have

$$\frac{d}{dt}\|RHS(t)\| = 2\|f(t)\|u_h(t)\| \leq 2\|f(t)\|\sqrt{\|RHS(t)\|},$$

and so

$$\frac{d}{dt}\sqrt{\|RHS(t)\|} \leq \|f(t)\|.$$

This implies that

$$\sqrt{\|RHS(t)\|} \leq \sqrt{\|RHS(0)\|} + \int_0^t \|f(s)\|ds,$$

So that

$$\|u_h(T)\| \leq \|u(0)\| + \int_0^T \|f(s)\|ds.$$

Therefore,the solution (4.1) is stable.

4.4 Error estimate

In this section, we will derive an error estimate from spatial FE discretization.

To do so, we use Ritz projection $R_h : H_0^1 \to V_h$ defined by:

$$B(R_h v, \chi) = B(v, \chi) \text{ for } v \in H_0^1 \text{ and } \chi \in V_h. \tag{4.5}$$

We have the well known approximation property:

$$\|R_h v - v\| \leq Ch^2 \|v\|_2 \text{ for } v \in H^2(\Omega) \cap H_1^0(\Omega). \tag{4.6}$$

The solution of the continuous problem (1.1) satisfies

$$\langle u^{'}(t), \chi\rangle + B(D^{\alpha}u(t), \chi) = \langle f(t), \chi\rangle \forall \chi \in H_0^1. \tag{4.7}$$

We divide the error into two terms as follows:

$$e(t) = u_h(t) - u(t) = \theta(t) + \rho(t),$$

where

$$\theta(t) = u_h(t) - R_h u(t) \text{ and } \rho(t) = R_h u(t) - u(t).$$

From Ritz projection we have this relation, $A\rho(t) = 0$.

From (4.1), (4.7), and definition of Ritz projection, we have

$$\langle \theta_t(t), \chi\rangle + B(D^{\alpha}\theta(t), \chi) = \langle -\rho_t(t), \chi\rangle.$$

By using stability, we have

$$\|\theta(t)\| \leq \theta(0) + 2\int_0^t \|\rho_t(s)\| ds. \tag{4.8}$$

Next we will bound the right-hand side of the above inequality. From the Ritz projection, we have

$$B((R_h u(t)_t) - R_h u(t))_t), \chi) = 0.$$

Choose $\chi = (R_h u(t))_t) - R_h u_t(t)$. Then we have, $\|(R_h u(t))_t - R_h u_t(t)\| = 0$. So $(R_h u(t))_t = R_h u_t(t)$. Therefore, $\rho_t(t) = (R_h u(t))_t - u_t(t) = R_h u_t(t) - u_t(t)$. By using (4.6) we have $\|\rho_t(t)\| \leq ch^2 \|u_t(t)\|$.

Using the above result the equation (4.8) becomes,

$$\|\theta(t)\| \leq \|\theta(0)\| + ch^2 \int_0^t \|u_t(s)\|_2 ds,$$

and so

$$\|u_h(t) - u(t)\| \leq \|\theta(t)\| + \|\rho(t)\| \leq ch^2 (\int_0^t \|u_t(s)\|_2 ds + \|u(t)\|_2).$$

By using fundamental of theorem of calculus and energy norm on $\mathsf{u}(t)$, we have

$$\|u(t)\|_2 \leq \int_0^t \|u_t(s)\|_2 ds + \|u(0)\|_2.$$

Therefore,

$$\|u_h(t) - u(t)\| \leq \|u_h(0) - R_h u(0)\| + Ch^2 (\int_o^t \|u_t(s)\|_2 \, ds + \|u(0)\|_2). \qquad (4.9)$$

CHAPTER 5

FULL DISCRETE SCHEMES

In this chapter we combine the time steeping FD schemes in chapter 3 with the spatial FEs in chapter 4. This will define fully discrete schemes.

5.1 Finite element, Backward Euler scheme

We combine backward Euler method in time (3.4) with linear finite element space (4.3) to get the following fully scheme. We will use equation (3.4) to get full scheme.

Talking the inner product of (3.4) with Φ_j. This gives,

$$\begin{aligned}\langle (I+\omega_{nn}A)U_h^n, \Phi_j\rangle &= \langle AU_h^{n-1}, \Phi_j\rangle + \langle f^n, \Phi_j\rangle \\ &\quad - \sum_{j=1}^{n-1}\langle \omega_{nj}AU^j, \Phi_j\rangle, j = 1, 2, ..., M_h,\end{aligned}$$

where

$$\beta^n = (\beta_i^n)_{M_h\times 1}, M = (\Phi_i, \Phi_j)_{M_h\times M_h}, S = (a\nabla\Phi_i\nabla\Phi_j)_{M_h\times M_h},$$

$$F(t) = (f(t), \Phi_j)_{M_h \times 1}, F_1(t) = (\tilde{f(t)}^1, \Phi_j)_{M_h \times 1}.$$

After rearranging the terms then we get the full scheme

$$\beta^n = [M + S\omega_{nn}]^{-1}[M\beta^{n-1} - \omega_{nn}M\beta^{n-1} + F^n - \sum_{j=1}^{n-1} \omega_{nj} S\beta^j].$$

5.2 Finite element, Crank-Nicolson scheme

We combine Crank-Niclson method in time (3.10) with linear finite element space (4.3) to get the following fully scheme. We will use equation (3.10) to get full scheme.

Taking the inner product of (3.10) with Φ_j(basis of V_h). This gives,

$$\begin{aligned}\langle (2I + \omega_{nn}A)U_h^n, \Phi_j\rangle &= \langle (2I - \omega_{nn}A)U_h^{n-1}, \Phi_j\rangle + 2\langle f^n, \Phi_j\rangle \\ &\quad - \sum_{j=1}^{n-1} \langle \omega_{nj} A(U^{j-1} + U^j), \Phi_j\rangle, j = 1, 2, ..., M_h,\end{aligned}$$

where

$$\beta^n = (\beta_i^n)_{M_h \times 1}, M = (\Phi_i, \Phi_j)_{M_h \times M_h}, S = (a\nabla\Phi_i, \nabla\Phi_j)_{M_h \times M_h}$$

$$F(t) = (f(t), \Phi_j)_{M_h \times 1}, F_1(t) = (\tilde{f(t)}^1, \Phi_j)_{M_h \times 1}.$$

After rearranging the terms then we get the full scheme.

$$\beta^n = [2M + S\omega_{nn}]^{-1}[2M\beta^{n-1} - \omega_{nn}S\beta^{n-1} + 2F^n - \sum_{j=1}^{n-1}\omega_{nj}S[\beta^{j-1} + \beta^j].$$

5.3 Error estimate

In this section we obtain an error estimate for the error $e^n = U_h^n - u(t_n)$ in the L_2 norm, where U^n and u are the solutions of (3.9)(or (3.3)) and (1.1) respectively. We divide the error into two terms as follows:

$$e^n = U_h^n - u(t_n) = \theta^n + \rho(t_n),$$

where

$$\theta^n = U_h^n - R_h u(t_n) \text{ and } \rho(t) = R_h u(t) - u(t).$$

Integrating (1.1) from $t = t_{n-1}$ to $t = t_n$ shows the exact solution u satisfies

$$u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} D^\alpha(Au)(t)dt = \tilde{f}^n. \tag{5.1}$$

(1) Error estimate for FEs in space and Crank-Niclson in time

From fully discrete we have

$$U_h^n - U_h^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha(A\bar{U}_h)(t)dt = \tilde{f}^n. \tag{5.2}$$

Comparing (5.1) and (5.2), we observe that the error e^n satisfies

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{U}_h - Au)(t)dt = \tilde{f}^n.$$

From definition, we have this $\bar{U}_h = \bar{e} + \bar{u}$, then inserting this in the above equation

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{e})(t)dt = \int_{t_{n-1}}^{t_n} D^\alpha (Au - A\bar{u})(t)dt. \tag{5.3}$$

From definition we have these

$$\bar{e} = \bar{\theta} + \bar{\rho}, \text{ and } e^n - e^{n-1} = (\theta^n - \theta^{n-1}) + (\rho(t_n) - \rho(t_{n-1})).$$

Equation (5.3) becomes

$$(\theta^n - \theta^{n-1}) + (\rho(t_n - \rho(t_{n-1}) + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{\theta} + A\bar{\rho})(t)dt = \int_{t_{n-1}}^{t_n} D^\alpha (Au - A\bar{u})(t)dt. \tag{5.4}$$

From the definition of the Ritz projector, we have this relation

$$A\bar{\rho} = 0.$$

Then, equation (5.3) becomes

$$(\theta^n - \theta^{n-1}) + \int_{t_{n-1}}^{t_n} D^\alpha (A\bar{\theta})(t)dt = \int_{t_{n-1}}^{t_n} D^\alpha (Au - A\bar{u})(t)dt - \rho(t_n) + \rho(t_{n-1}). \tag{5.5}$$

We apply inner product χ in the equation (5.5), then we get

$$\langle \theta^n - \theta^{n-1}, \chi\rangle + \int_{t_{n-1}}^{t_n} \langle D^\alpha (A\bar{\theta})(t), \chi\rangle dt = \langle \eta^{j-\frac{1}{2}}, \chi\rangle.$$

By using stability property. We have

$$\|\theta^n\| \leq \|U_h^0 - R_h u_0\| + 2\sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\|.$$

We make the splitting

$$\hat{\eta}^{j-\frac{1}{2}} = \eta^{j-\frac{1}{2}} + \hat{\eta}_1^{j-\frac{1}{2}} + \hat{\eta}_2^{j-\frac{1}{2}},$$

$\eta^{j-\frac{1}{2}}$ the same as in (3.12) and $\hat{\eta}_1^{j-\frac{1}{2}} + \hat{\eta}_2^{j-\frac{1}{2}}$ the same as in (4.9) . By using this we get the desired result.

(2) **Error estimate for FEs in space and Backward Euler in time**

We use the same technique of (1) by replacing $^-$ with $^\rightarrow$ and using the stability property of

$$\|\theta^n\| \leq \|U_h^0 - R_h u_0\| + \sum_{j=1}^{n} \|\eta^{j-\frac{1}{2}}\|.$$

Then we have $\eta^{j-\frac{1}{2}}$ which is the same as (3.6) and $\hat{\eta}_1^{j-\frac{1}{2}} + \hat{\eta}_2^{j-\frac{1}{2}}$ which is the same as (4.9) . By using this we get the desired result.

5.4 Numerical Results

In this section we provide numerical experiments which should enable a better understanding of our theoretical results.

Example 5.1 *Let* $\Omega = (0, 1)$ *and* $Au = -u_{xx}$ *and assume that* $u = u(x, t)$ *satisfies homogeneous Dirichlet boundary conditions* $u(0, t) = 0 = u(1, 0)$ *for all* $t \in [0, T] = [0, 1]$*. Choosing the initial datum* $u_0 = -\frac{\Gamma(2-\alpha)}{\pi^2}\sin(\pi x)$ *and a source term* $f(x, t) = \frac{\Gamma(2-\alpha)}{\Gamma(2-2\alpha)}\pi^2 t^{1-2\alpha} sin(\pi x)$*. This problem has exact solution*

$$u(x) = (t^{1-\alpha} - \frac{\Gamma(2-\alpha)}{\pi^2})\sin(\pi x).$$

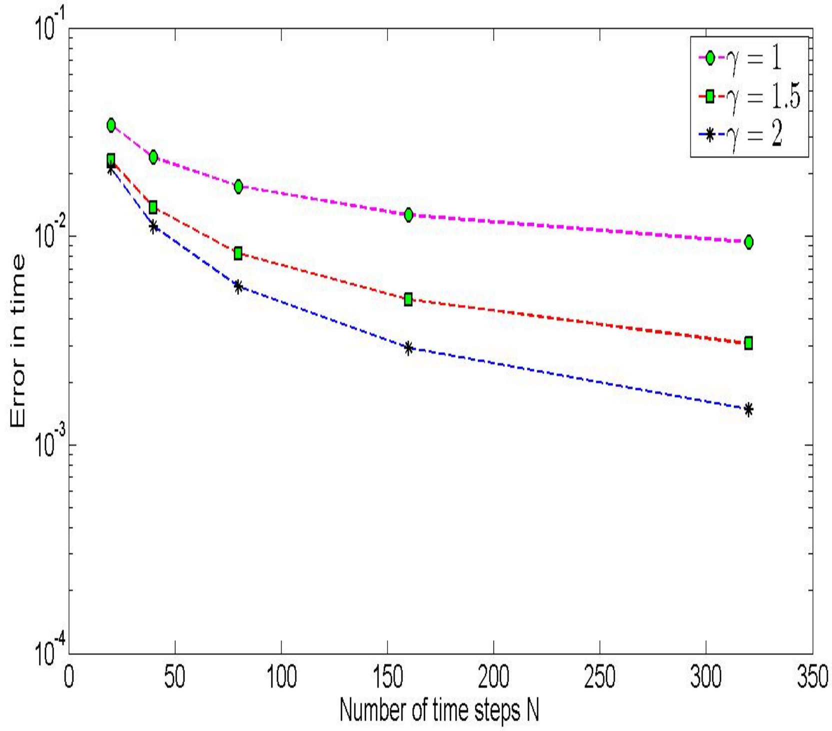
The regularity condition (2.3) holds for $\sigma = 2 - 2\alpha$ and next we will see our theoretical result with numerical result.

M.Error= Maximum nodal time error OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	3.4179e-02	–	2.3054e-02	–	2.1266e-02	–
40	2.3943e-02	0.514	1.3757e-02	0.745	1.1143e-02	0.932
80	1.7347e-02	0.465	8.2540e-03	0.737	5.7338e-03	0.959
160	1.2653e-02	0.455	4.9889e-03	0.726	2.9209e-03	0.973
320	9.3624e-03	0.435	3.0449e-03	0.712	1.4788e-03	0.982
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	7.1074e-02	–	2.2485e-02	–	6.2017e-03	–
40	4.2308e-02	0.748	9.3111e-03	1.272	1.9626e-03	1.66
80	2.4494e-02	0.788	3.6734e-03	1.342	6.9315e-04	1.501
160	1.3794e-02	0.828	1.3959e-03	1.396	2.4533e-04	1.498
320	7.5746e-03	0.864	5.1666e-04	1.434	8.6790e-05	1.499

Table 5.1: *Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.5$. We observe better orders for the CN scheme. The errors and convergence rates for the BW and CN improved when the mesh is graded. We observe convergence rate of order for BW k^1 for $\gamma > \frac{1}{\sigma}$ and $k^{\gamma\sigma}$ for $\gamma < \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.*

Backward Euler



Crank-Nicolson

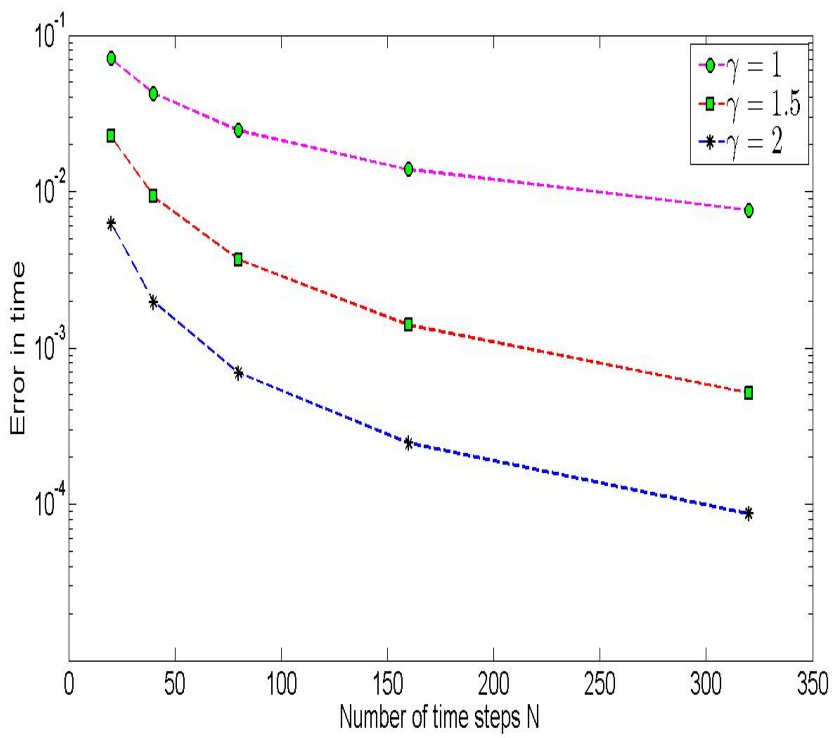


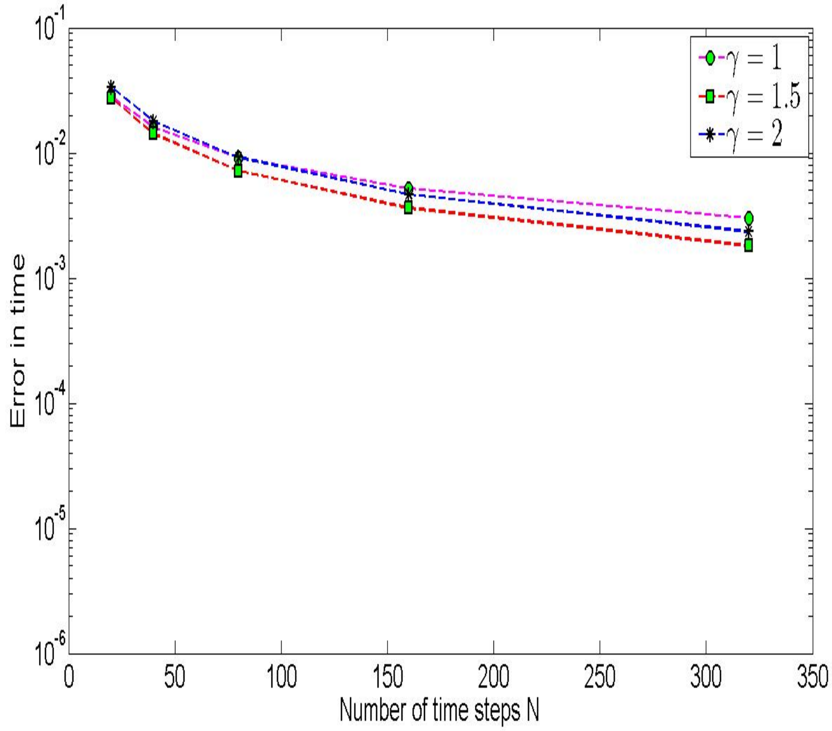
Figure 5.1: *Error against time steps* $\alpha = 0.5$.

M.Error= Maximum nodal time error   OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	2.8500e-02	–	2.7658e-02	–	3.3427e-02	–
40	1.6071e-02	0.827	1.4198e-02	0.962	1.7793e-02	0.910
80	9.0689e-03	0.825	7.1868e-03	0.982	9.1708e-03	0.956
160	5.1753e-03	0.809	3.6140e-03	0.992	4.6560e-03	0.978
320	2.9955e-03	0.789	1.8115e-03	0.996	2.3463e-03	0.989
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	6.3930e-03	–	1.1598e-03	–	1.1843e-03	–
40	2.4167e-03	1.403	3.4547e-04	1.747	3.5082e-04	1.755
80	8.7491e-04	1.466	1.0263e-04	1.751	1.0352e-04	1.761
160	3.0639e-04	1.514	3.0390e-05	1.756	3.0440e-05	1.766
320	1.0489e-04	1.547	8.9600e-06	1.761	8.9200e-06	1.77

Table 5.2: *Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.2$. We observe better orders for the CN scheme. The errors and convergence rates for CN improved when the mesh is graded. We observe convergence rate of order for BW k^1 for $\gamma > \frac{1}{\sigma}$ and $k^{\gamma\sigma}$ for $\gamma < \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.*

Backward Euller



Crank-Nicolson

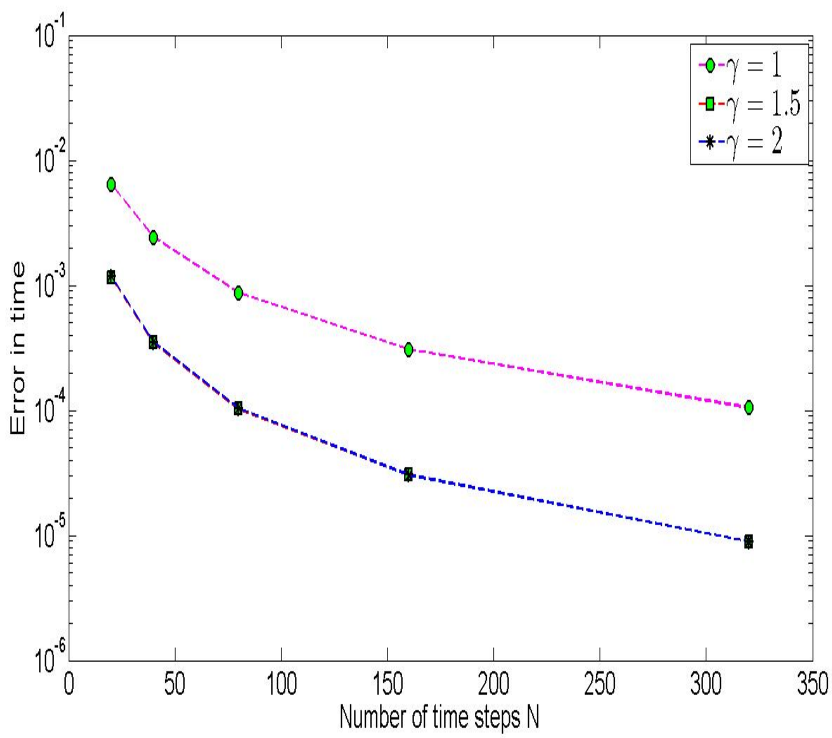


Figure 5.2: *Error against time steps for $\alpha = 0.2$.*

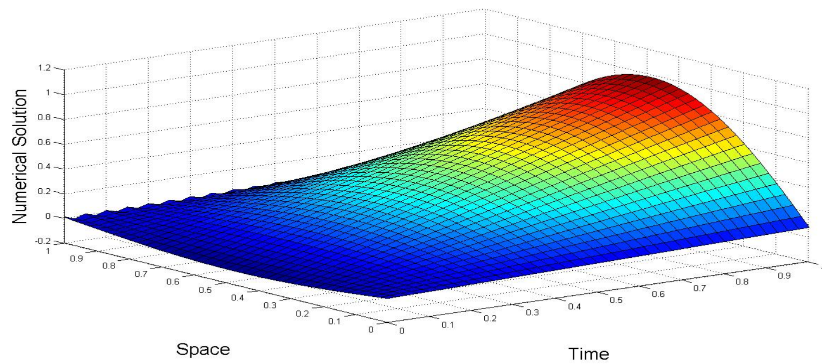


Figure 5.3: *Numerical solution for N=40,* $\gamma = 2$ *and* $\alpha = 0.5$*.*

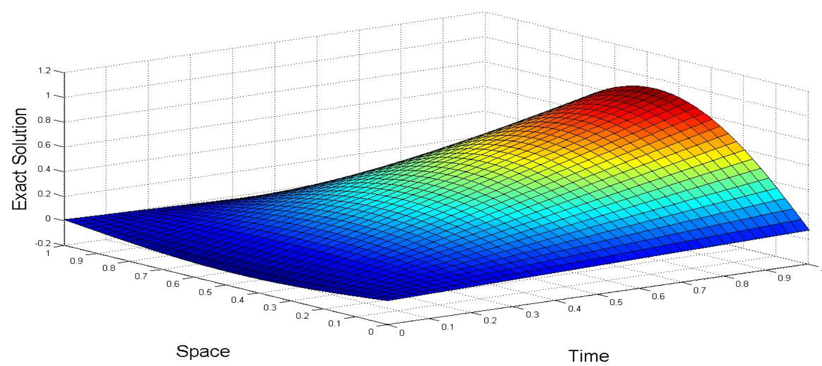
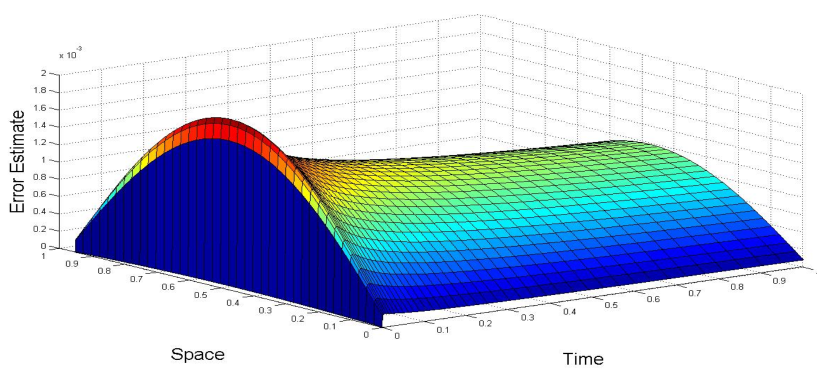


Figure 5.4: *Exact solution for a given problem.*



1.jpg

Figure 5.5: *Error estimate for N=40,* $\gamma = 2$ *and* $\alpha = 0.5$*. .*

Example 5.2 (a less regular solution)

Let $u_0(x) = x(1-x)$ and $f = 0$. Thus, $Au_0 = 2$ does not vanish at 0 and 1, so $u_0 \notin D(A)$. The exact solution is

$$\sum_{n=0}^{\infty} w_n^{-3} sin(w_n x) E_{1-\alpha}(-w_n^2 t^{1-\alpha}) \text{ with } w_n = (2n+1).$$

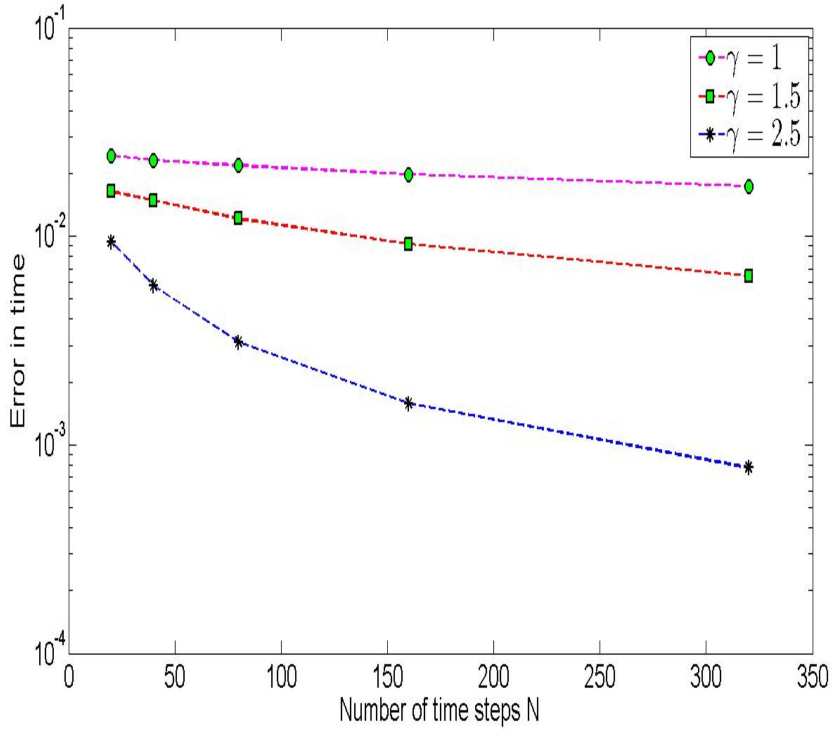
The regularity condition (2.3) holds for $\sigma = 1.25(1-\alpha)$ and next we will see our theoretical result with numerical result.

M.Error= Maximum nodal time error OC=Order of Convergence

Backward Euler						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2.5$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	2.4336e-02	–	1.6403e-02	–	9.4085e-03	–
40	2.3216e-02	6.79e-02	1.4888e-02	0.140	5.7954e-03	0.699
80	2.1793e-02	9.13e-02	1.2153e-02	0.293	3.1209e-03	0.893
160	1.9770e-02	0.141	9.1732e-03	0.406	1.5782e-03	0.984
320	1.7358e-02	0.188	6.4633e-03	0.505	7.7738e-04	1.02
Crank-Nicolson						
	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2.5$	
N	M.Error	OC	M.Error	OC	M.Error	OC
20	8.2191e-02	–	3.4805e-02	–	5.7777e-03	–
40	5.7730e-02	0.51	1.6867e-02	1.045	2.319e-03	1.317
80	3.7574e-02	0.62	9.0270e-03	0.901	9.7233e-04	1.254
160	2.2980e-02	0.709	5.1924e-03	0.798	4.0822e-04	1.252
320	1.4525e-02	0.662	3.0700e-03	0.758	1.7156e-04	1.251

Table 5.3: *Order of convergence and maximum error with different mesh grading γ with $\alpha = 0.5$. We observe better orders for the CN scheme. The errors and convergence rates for the BW and CN improved when the mesh is graded. We observe convergence rate of order for BW k^1 for $\gamma > \frac{1}{\sigma}$ and $k^{\gamma\sigma}$ for $\gamma < \frac{1}{\sigma}$. For CN we observe $k^{\gamma\sigma}$ for $\gamma < \frac{2-\alpha}{\sigma}$ and $k^{2-\alpha}$ for $\gamma > \frac{2-\alpha}{\sigma}$.*

Backward Euler



Crank-Nicolson

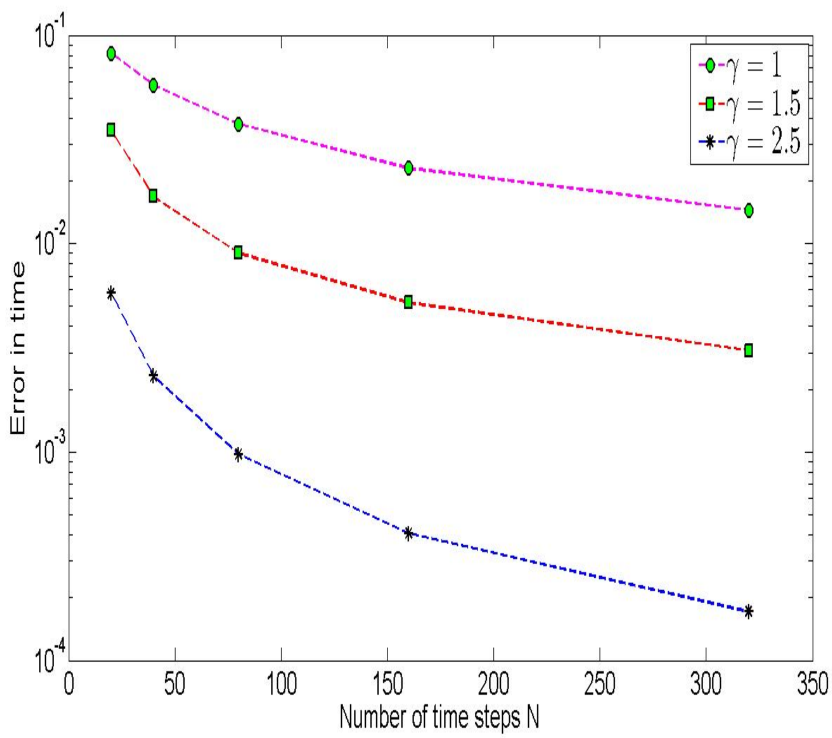


Figure 5.6: *Error against time steps for* $\alpha = 0.5$.

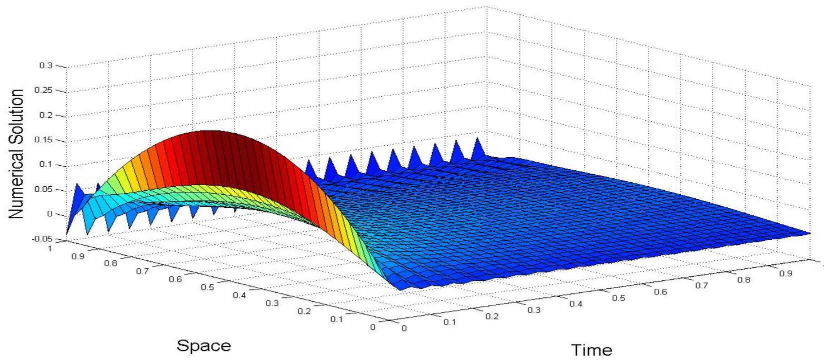


Figure 5.7: *Numerical solution for N=40,* $\gamma = 2.5$ *and* $\alpha = 0.5$.

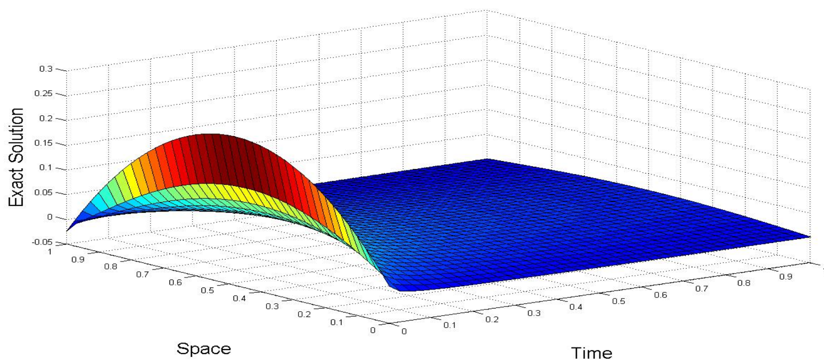


Figure 5.8: *Exact solution for a given problem.*

2.jpg

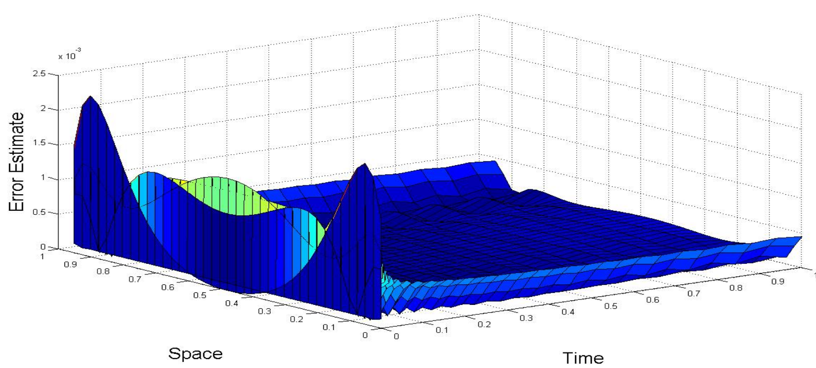


Figure 5.9: *Error estimate for N=40,* $\gamma = 2.5$ *and* $\alpha = 0.5$. .

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